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Efficiency, Quality of Forecasts and Radner Equilibria

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Abstract

We study a simple two period economy with no uncertainty and complete markets where agents trade based on forecasts about the second period spot price. We propose as our solution concept a *set* of forecasts with the following properties: there exist (heterogenous) forecasts contained in this set that lead to efficient allocations, the set contains only those forecasts that correspond to some efficient equilibrium, and finally that the forecasts assign positive probability to the actual market clearing spot price. We call such a set of prices an *efficient equilibrium with ambiguity*, and interpret it as a generalization of Radner equilibrium that delivers efficient allocations under forecasts that possess a self-fulfilling property that is weaker than perfect foresight.

JEL classification numbers : D51, D53, D61

1 Introduction

Walrasian trade in intertemporal economies require households to forecast prices that will prevail in spot markets at future dates. The ubiquitous financial equilibrium model that is used to address this aspect of intertemporal economies is the one proposed by Radner (1972) (following Arrow (1963)) and is the bedrock of modern treatments of

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general equilibrium. This resulting Radner equilibrium (henceforth, RE), postulates that households correctly anticipate all spot prices at future dates; a RE is accordingly an equilibrium with perfect foresight (henceforth, PFE), where the forecasts of heterogenous households are perfectly aligned.

Consider a simple two period environment with no uncertainty. An important implication of perfect foresight is that with one financial asset, say a nominal bond, the market structure is equivalent to a model with complete markets. Therefore, a RE delivers an efficient allocation. Earlier work (Chatterji, Kajii and Zeng (2018a, 2018b), Chatterji and Kajii (2020)) has shown that the efficiency of equilibrium does not dictate that forecasts have to perfectly aligned and self-fulfilling, that is, perfect foresight is not an implication of efficiency. These papers demonstrate that with time separable utilities, generically in endowments, there exists a one-dimensional set of efficient temporary equilibrium allocations (henceforth, ETE) around each PFE which are supported by heterogenous forecasts. These results indicate that requiring that markets allocate resources efficiently does not in any way pin down perfect foresight as the only configuration of forecasts that is consistent with the efficiency of equilibrium.

From another perspective, we may interpret these results as saying that the hypothesis that markets allocate resources efficiently, fails to provide a foundation for the postulate of perfect foresight, and hence for RE. However, in the aforementioned works, those forecasts that are implicit in supporting the one dimensional set of efficient allocations, may be rather disparate across households and need not satisfy any self-fulfilling property[1]; this leaves the possibility that requiring some weak form of the self-fulfilling property of the forecasts that are used in supporting an ETE may lead one all the way to perfect alignment of forecasts, that is, all the way to perfect foresight. If this were true, efficiency of markets along with some possibly weak requirement on the quality of forecasts would pin down perfect foresight and provide thereby a foundation to RE.

Our investigation of a simple two period model with one good in each period reveals that in general such a conclusion is not warranted. We proceed by proposing as our solution concept a *set* of price forecasts with the following property: every household's

[1] In the Chatterji and Kajii (2020) paper, households may agree, and be correct, on the relative prices being forecasted and disagree solely on the rate of inflation.

forecast is composed only of prices that correspond to efficient equilibria. Moreover, such a set of prices can be interpreted as a generalization of PFE to an *efficient equilibrium with ambiguity*, which proposes a set of efficient equilibria that possess a self-fulfilling property which is weaker than perfect foresight.

Notice that the set of all PFE of an economy trivially satisfy the requirements of this proposed set. The contribution of this paper is then to show that there exists a robust non-trivial case of economies where the proposed set differs from the trivial ones. We identify a condition on the slope of the local forecast functions (that are implied by the analysis of the aforementioned papers) evaluated at the PFE, that ensures that one may construct an interval of prices around a PFE that satisfies the requirements of our proposed set. In particular, this interval contains non PFE prices that are consistent with some efficient equilibrium.

To summarize, the one dimensional set of efficient equilibria around a PFE identified in earlier work can be supported by a set of prices that possess a particular form of a self-fulfilling property.

The remainder of the paper is organized as follows. Section 2 describes the model, reviews RE and efficient equilibria. Section 3 notes a weak self-fulfilling property of the efficient equilibria while Section 4 proposes our notion of efficient equilibrium with ambiguity and illustrates using an example. Section 5 concludes.

2 The Model and Definition

2.1 Setup

We consider an exchange economy as follows. There are two periods, period 0 and 1, and there is one perishable consumption good in each period. We simply call the good of the first period good 0 and the good of the second period good 1.

There are $H \geq 1$ households, labelled by $h = 1, ..., H$. Abusing notation we use H for the set of households as well. Household h is endowed with e_h^0 units of good in the first period (period 0) and e_h^1 units in the second period (period 1). We write $e_h = \left(e_h^0, e_h^1\right)$. Household h's consumption set is $\mathbb{R}_+^2$ with a generic element denoted by $\left(x^0, x^1\right)$ and its preferences for consumption bundles are represented by an increasing, continuous, and strictly concave utility function $u_h : \mathbb{R}^2_+ \to \mathbb{R}$. We will allow random consumption and households are assumed to be expected utility maximizers to evaluate random consumption vectors. That is, if $\tilde{x}^1$ is a non-negative random variable, the utility from $\left(x^0, \tilde{x}^1\right)$ is $\mathbf{E}\left[u_h\left(x^0, \tilde{x}^1\right)\right]$,

Set $\bar{e}^t = \sum_{h=1}^H e_h^t$, for $t = 0, 1$; that is, $\bar{e}^t$ is the total supply of the good t, and we assume $\bar{e}^t > 0$ for both t. An allocation of goods, $\left(x_h^0, x_h^1\right)_{h=1}^H \in \left(\mathbb{R}^2_+\right)^H$, is feasible if $\sum_{h=1}^H x_h^t = \bar{e}^t$ for $t = 0, 1$. When consumption is random, feasibility requires the equality holds with probability one.

A feasible allocation is said to be Pareto efficient if there is no alternative feasible allocation which improves all households utility level. With a strict concave utility function, every household is strictly risk averse, and hence an efficient allocation must necessarily be non-random since there is no uncertainty in the aggregate. Moreover, when $\bar{e}^0 = \bar{e}^1$, a feasible allocation is efficient if and only if every household consumes the same amount in both periods, i.e., perfect consumption smoothing occurs.

Since we borrow the results from earlier papers which employ the standard technique of genericity analysis, we assume in addition the following , although some of them are not explicitly invoked in this paper: for every household $h = 1, ..., H$,

- utility function u_h is C^2 on $\mathbb{R}^2_{++}$, $\partial u_h \gg 0$, and differentiably strictly concave, and each indifference curve is closed in $\mathbb{R}^2$;
- initial endowments e_h are strictly positive.

Furthermore, we fix utility functions throughout, and identify an economy with its initial endowments: so write $\mathcal{E} := \left(\mathbb{R}^2_{++}\right)^H$ and its generic element is denoted by $e = (\cdots, e_h, \cdots)$. We say a subset of $\mathcal{E}$ is *generic* if it is open and its complement has Lebesgue measure 0.

2.2 Temporary Equilibrium and Radner equilibrium

We begin by describing a model of perfect competition that does not impose a common and correct forecast a priori.

In period 0, a riskless bond which pays off 1 dollar in period 1 is traded competitively. There is no default and no borrowing constraint. We may assume without loss of generality that the bond price is equal to the price of good 0, which is equal to one by normalization. Writing z_h for the amount of the bond held by household h, the choice of consumption and saving is therefore subject to $e_h^0 - x_h^0 = z_h$ in period 0.

A forecast for the period 1 price is a positive random variable. Denote the set of all forecasts by $\mathcal{F}$. With a forecast $\tilde{p}_h \in \mathcal{F}$, household h expects that the period 1 budget will be $\tilde{p}_h \left(x_h^1 - e_h^1\right) \leq z_h$. Since we have assumed strict monotonicity, all budget constraints will be satisfied as equalities and thus his random consumption will be $\tilde{x}_h^1 = e_h^1 + z_h/\tilde{p}_h$ with probability one. That is, he will choose x_h^0 anticipating such a random consumption $\tilde{x}_h^1$. Equivalently, by elimination of z_h, household h chooses x_h^0 associated with a utility maximizing random consumption $\left(x_h^0, \tilde{x}_h^1\right)$ which satisfies the following budget constraint with probability one:

$$\left(x_h^0 - e_h^0\right) + \tilde{p}_h \left(\tilde{x}_h^1 - e_h^1\right) = 0. \tag{1}$$

With a deterministic forecast, i.e., $\tilde{p}_h = \hat{p}_h$ with probability one, the constraint may be written as

$$\left(x_h^0 - e_h^0\right) + \hat{p}_h \left(\hat{x}_h^1 - e_h^1\right) = 0, \tag{2}$$

where $\hat{x}_h^1 = e_h^1 + z_h/\hat{p}_h$.

We denote the realized market price of the good in period 1 by p, and then the realized consumption path $\left(x_h^0, x_h^1\right)$ must satisfy the following equation:

$$\left(x_h^0 - e_h^0\right) + p \left(x_h^1 - e_h^1\right) = 0. \tag{3}$$

Of course, if $\tilde{p}_h = p$ with probability one, two budget equations (1) and (3) are identical and so are the respective consumption choices. We write $\tilde{p}$ for a profile of forecasts, i.e., $\tilde{p} = (\cdots, \tilde{p}_h, \cdots)$. A temporary equilibrium consists of a profile of consumption bundles, a profile of forecasts, and a market clearing price of the second period, formally defined as follows:

Definition 1 *A temporary equilibrium (TE) is a tuple* $(x^*, \tilde{p}, p^*) \in \left(\mathbb{R}^2_{++}\right)^H \times (\mathcal{F})^H \times \mathbb{R}_+$ *such that:*

(i) x^* *is a feasible allocation, i.e.,* $\sum_{h=1}^H x_h^* = \sum_{h=1}^H e_h$*;*

(ii) for each $h \in H$*, there exists a random variable* $\tilde{x}_h^1$ *such that* $\left(x_h^{0*}, \tilde{x}_h^1\right)$ *maximizes utility under budget (1) given forecast* $\tilde{p}_h$*;*

(iii) for each $h \in H$, x_h^{1*} maximizes $u_h\left(x_h^{0*}, \cdot\right)$ under constraint (3) at $p = p^*$, and $x_h^0 = x_h^{0*}$.

A temporary equilibrium is said to be a perfect foresight equilibrium (PFE) if $\tilde{p}_h = p^*$ holds with probability one for every household h

Given a TE $(x^*, \tilde{p}, p^*)$, x^* is called a *TE allocation* and p^* is called a *TE price,* which by definition clears the second period market. We refer to condition (ii) above as *justifiability*: we say a forecast $\tilde{p}_h$ justifies consumption x_h^{0*} if (ii) holds for household h, since it means that the period 0 consumption can be explained by utility maximization under the forecast. Since forecasts appear in (ii) only, if another forecast $\tilde{p}'_h$ justifies the same period 0 consumption for each h, $(x^*, \tilde{p}', p^*)$ constitutes a TE. It can be readily seen that there is a large variety of random forecasts which justifies the same consumption, and hence there is a large indeterminacy about forecasts for a TE $(x^*, \tilde{p}, p^*)$.

With slight abuse of notation, when $\tilde{p}_h = \hat{p}_h \in R_+$ with probability one for every h, i.e., every household has a point forecast, we write $(x^*, \hat{p}, p^*)$ where $\hat{p} = (\hat{p}_h)_{h=1}^H$, and we shall further simplify it to (x^*, p^*) when $\hat{p} = p^*$, i.e., every household's point forecast agrees with the market price, to economize notation. In particular, note that anticipated consumption $\tilde{x}_h^1$ is non random in a PFE, and must coincide with x_h^{1*}. Thus in effect, if a TE is a PFE, every household anticipates the future consumption correctly and so a PFE will be written as (x^*, p^*). Since the two constraints (1) and (3) coincide at a PFE, it is readily seen that a PFE is equivalent to an Arrow Debreu equilibrium where two goods are simultaneously traded.

The PFE is an instance of a Radner Equilibrium (henceforth RE) which, as is well known, is defined more generally so as to apply to incomplete market settings as well. The key feature of a RE is that rational expectations is assumed rather than derived, and hence the concept hinges on the exact coordination of household forecasts at particular 'self-fulfilling' price, which in our simple setting, is equivalent to rational expectations. Our intention here is to re-examine the content of the assumption of rational expectations, and to that end we propose a more permissive equilibrium notion than a RE that allows heterogenous forecasts while retaining the self-fulfilling feature of forecasts in a somewhat weaker form.

The underlying postulate in our analysis is that dynamic trading activities will take place until gains from trade are exhausted. In other words, just as perfect competition based on price taking behavior is intended to be an “as-if” story to formalize anonymous and voluntary trading of many economic agents, we seek such an as-if story to formalize the exhaustion of gainful trading opportunities. Towards this purpose, instead of imposing rational expectations, we impose rather the efficiency of the resulting allocation: our notion will draw upon the idea of an efficient temporary equilibrium.

2.3 Efficient Temporary Equilibria

Let p^* be a PFE price and x^* be the associated allocation, and assume $x_h^* \neq e_h$ for every h. It can be shown[2] that such PFE exist and are locally unique, generically in endowments under our assumptions.

Note that by revealed preference, $u_h(x_h^*) > u_h(e_h)$ for every h. That is, a PFE is *individually rational*. On the other hand, a TE might not be individually rational, since households do not necessarily anticipate the correct constraint. Note however that a temporary equilibrium also exhibits individual rationality if the resulting allocation is close enough to such a PFE allocation. Thus we shall focus mostly on TE close to a PFE.

A PFE is *self-selective* in the sense that no household prefers a net trade of another household to its own. This is simply a consequence of utility maximization on a common set of budget feasible net trade. A TE still exhibits similar self-selectiveness at their respective forecasts: no household prefers a saving/borrowing position of another household, given its forecasts.

By the first fundamental theorem of welfare economics, a PFE allocation is an efficient allocation. Given our postulate, a natural preliminary question to ask here is if there is an efficient TE other than PFE. The answer turns out to be generically yes.

Definition 2 *An efficient temporary equilibrium (ETE) is a temporary equilibrium* $(x^*, \tilde{p}, p^*)$ *where the consumption allocation* x^* *is Pareto efficient.*

Under our assumptions, Chatterji, Kajii and Zeng (2018b) demonstrate essentially

[2] For instance, Mas-Colell (1985).

the following [3]:

Proposition 3 *(Chatterji, Kajii and Zeng (2018b)) There is a generic set $\mathcal{E}^* \subset \mathcal{E}$ such that for each $e \in \mathcal{E}^*$, (i) there are finitely many PFE, (ii) for each PFE allocation $\bar{x} \in \left(\mathbb{R}^2_{++}\right)^H$, there is a one dimensional C^1 manifold of ETE allocations containing $\bar{x}$ parameterized by the ETE price p around the respective PFE price, and (iii) there is for each h, a unique deterministic forecast parameterized by the ETE price p which justifies the ETE allocation.*

ETE will serve as the basis for two notions of Self-fulfilling ETE that we now turn to.

3 Weakly Self-fulfilling Efficient Temporary Equilibrium

As we noted earlier, the precise connection between PFE and ETE will hinge on the quality of forecasts, that is, whether or not the forecasts justifying the households' period 0 consumption has some self-fulfilling quality.

Definition 4 *A profile of forecasts $\tilde{p}$ is weakly self-fulfilling at p if p belongs to the support of $\tilde{p}_h$, for every h.*

That is, a random forecast is weakly self-fulfilling if it does not rule out the price to be realized in the market. This seems to be the minimal consistency requirement for a reasonable forecast. In what follows, we argue that any ETE allocation with deterministic forecasts arises with weakly self-fulfilling random forecasts. Let $(x^*, \hat{p}, p^*)$ be an ETE where $\hat{p} = (\hat{p}_h)_{h=1}^H$ is a profile of *deterministic* forecasts. Note that forecasts matter only to the extent of justifying period 0 consumption for each household (i.e., in condition (ii) of Definition 1). Thus it suffices to construct a random forecast which assumes p^* with positive probability and which justifies the same period 0 consumption.

Given the large degree of freedom about random forecasts, we will focus on simple binary random forecasts which are weakly self-fulfilling.

[3] Their main result is not stated in this way exactly, but we shall omit a proof since this result can be readily established by a close examination of their analysis with a help of a more general analysis carried out in Chatterji and Kajii (2020).

Definition 5 *A profile of forecasts $\tilde{p}$ is said to be binary if the support of $\tilde{p}_h$ consists of at most two points for every h.*

It will be convenient (omitting the subscript h to simplify notation) to denote $v(a,b) := u\left(e^0 + a, e^1 + b\right)$. From the utility maximization problem of a household, let $\hat{z} = e^0 - x^0$ be the optimal saving at the deterministic forecast $\hat{p}$. Finally, let

$$k^{\hat{z}}(q) := -v_a(-\hat{z}, q\hat{z}) + v_b(-\hat{z}, q\hat{z})\, q. \tag{4}$$

That is, $k^{\hat{z}}(q)$ is the marginal utility from saving and $k^{\hat{z}}(q) = 0$ will constitute the relevant first order condition for consumption choice at price p, where $q = \frac{1}{p}$.

The following result, which asserts in addition that the random forecast takes a value closer to the original deterministic forecast, is stronger than is necessary in this context, but will turn out to be useful in the next section. To see how it works, notice that the expected marginal utility from saving at a random forecast is the expectation of marginal utilities for forecasted prices. So a household will save the same amount at a random forecast as it does at a deterministic forecast, if the expectation of marginal utilities is kept the same. Starting with an ETE $(x^*, \hat{p}, p^*)$ where $\hat{p} = (\hat{p}_h)_{h=1}^H$ is a profile of deterministic forecasts, we will modify the forecast so that the ETE price p^* occurs with a small positive probability. The expected marginal utilities would remain the same if the marginal utility from the deterministic forecast can be adjusted to offset the change induced by the ETE price. The Lemma below shows that unless the marginal utility of saving at the deterministic forecast is insensitive to a price change, such a construction can be done.

Lemma 6 *Suppose $\left(x^0, \hat{x}^1\right)$ maximizes utility under budget (2) given a deterministic forecast $\hat{p}$. Assume that $\frac{dk^{\hat{z}}(\hat{q})}{dq} \neq 0$ and let $p^* \neq \hat{p}$ be any price with $k^{\hat{z}}(q^*) \neq 0$, where $\hat{q} = 1/\hat{p}$ and $q^* = 1/p^*$.*

Then for any $\delta > 0$, there exists a non-degenerate random forecast $\tilde{p}$, which takes values p^* and p' where $|\hat{p} - p'| < \delta$, such that $\left(x^0, \tilde{x}^1\right)$ maximizes utility under budget (1).

Proof. Recall that $\hat{z} = e^0 - x^0$ is the optimal saving at the deterministic forecast $\hat{p}$. We shall seek a random forecast which assigns probability ε to p^* and $1 - \varepsilon$ to p' which is close to $\hat{p}$. For given $p' > 0$ and $\varepsilon \geq 0$, consider the problem of maximizing the objective function by choice of z, where $q' = 1/p'$, under such a random forecast:

$$u\left(e^0 - z, e^1 + \frac{z}{p'}\right)(1-\varepsilon) + u\left(e^0 - z, e^1 + \frac{z}{p^*}\right)\varepsilon = v\left(-z, q'z\right)(1-\varepsilon) + v\left(-z, q^*z\right)\varepsilon$$

It is a concave problem, and the first order necessary and sufficient condition is

$$\phi\left(z; q', \varepsilon\right) := \left(-v_a\left(-z, q'z\right) + v_b\left(-z, q'z\right)q'\right)(1-\varepsilon) + \left(-v_a\left(-z, q^*z\right) + v_b\left(-z, q^*z\right)q^*\right)\varepsilon = 0,$$

where v_a and v_b are the respective partial derivatives. Notice that $\phi\left(z; q', 0\right) = k^{\hat{z}}(q')$ by construction.

By hypothesis, $\hat{z}$ is an optimal choice when $(q', \varepsilon) = (\hat{q}, 0)$, so we have

$$\begin{aligned}\phi\left(\hat{z}; \hat{q}, 0\right) &= -v_a\left(-\hat{z}, \hat{q}\hat{z}\right) + v_b\left(-\hat{z}, \hat{q}\hat{z}\right)\hat{q} \\ &= 0,\end{aligned}$$

and are done if we can find (q', ε) with $q' > 0$ and $0 < \varepsilon < 1$ such that $\phi\left(\hat{z}; q', \varepsilon\right) = 0$.

Differentiating $\phi\left(\hat{z}; q', \varepsilon\right)$ with respect to ε and q' respectively, and evaluating these at $\varepsilon = 0$ and $q' = \hat{q}$, recalling that $k^{\hat{z}}(q') \equiv -v_a\left(-\hat{z}, q'\hat{z}\right) + v_b\left(-\hat{z}, q'\hat{z}\right)q' = \phi\left(\hat{z}, q', 0\right)$, and $k^{\hat{z}}(\hat{q}) = 0$, we find that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\phi\left(\hat{z}, \hat{q}, 0\right) &= -\left(-v_a\left(-\hat{z}, \hat{q}z\right) + v_b\left(-\hat{z}, \hat{q}z\right)\hat{q}\right) + \left(-v_a\left(-\hat{z}, q^*z\right) + v_b\left(-\hat{z}, q^*\hat{z}\right)q^*\right) \\ &= -k^{\hat{z}}(\hat{q}) + k^{\hat{z}}\left(q^*\right) \\ &= k^{\hat{z}}\left(q^*\right),\end{aligned}$$

and,

$$\frac{\partial}{\partial q'}\phi\left(\hat{z}, \hat{q}, 0\right) = \frac{dk^{\hat{z}}\left(\hat{q}\right)}{dq'}.$$

Our assumptions that $k^{\hat{z}}\left(q^*\right) \neq 0$ and $\frac{dk^{\hat{z}}(\hat{q})}{dq'} \neq 0$, allow us to apply the Implicit Function Theorem and obtain the local solution $\varepsilon\left(q'\right)$ around $\hat{q}$ that satisfies

$$\phi\left(\hat{z}; q', \varepsilon\left(q'\right)\right) = 0$$

and whose derivative is

$$\frac{\partial\varepsilon\left(\hat{z}, \hat{q}, 0\right)}{\partial q'} = -\frac{\frac{dk^{\hat{z}}(\hat{q})}{dq'}}{k^{\hat{z}}\left(q^*\right)} \neq 0.$$

If $\frac{\partial \varepsilon(\hat{z},\hat{q},0)}{\partial q'} > 0$ (resp, < 0), then for $q' > \hat{q}$ (resp, < 0) such that $|q' < \hat{q}|$ is small enough, we have $1 > \varepsilon(q') > 0$ as required to complete the proof. ■

For an ETE $(x^*, \hat{p}, p^*) \in (\mathbb{R}^2_{++})^H \times (\mathbb{R}_+)^H \times \mathbb{R}_+$, write $z^*_h \equiv e^0_h - x^{*0}_h$, and define function $k_h^{z_h^*}$ analogously to (4) for each household h. Then we establish the following:

Proposition 7 *Let $(x^*, \hat{p}, p^*) \in (\mathbb{R}^2_{++})^H \times (\mathbb{R}_+)^H \times \mathbb{R}_+$ be an ETE with deterministic forecasts. Suppose that for each household h, either $\hat{p}_h = p^*$ or $\frac{dk_h^{\hat{z}}(1/\hat{p}_h)}{dq} \neq 0$ holds. Then there is a profile of binary random forecasts, $\tilde{p}$, such that $(x^*, \tilde{p}, p^*)$ constitutes an ETE where for every h, $\tilde{p}_h$ is weakly self-fulfilling at p^*.*

Proof. If $\hat{p}_h = p^*$, just let $\tilde{p}_h$ be the deterministic forecast $\hat{p}_h$ itself. If $\hat{p}_h \neq p^*$ and $k^{\hat{z}}(1/p^*) = 0$, then household h's demand at the ETE is justified at p^* as well, so set $\tilde{p}_h$ be the deterministic forecast p^*. If $\hat{p}_h \neq p^*$ and $k^{\hat{z}}(1/p^*) \neq 0$, apply Lemma 6 to obtain a binary forecast $\tilde{p}_h$ which is weakly self-fulfilling. By construction, for each h, $\tilde{p}_h$ assigns strictly positive property to p^* as required. ■

4 Equilibrium with price ambiguity

In the elaboration of ETE thus far, households are required to have a forecast but the process of forecasting is not structured. The classical literature[4] assumes an exogenous ad hoc forecasting rule but that is not the path we would like to pursue. Instead, we propose a concept of two-step forecasting which can serve as a particular formulation of self-fulfilling expectations.

Specifically, we assert that each household first identifies a set of possible market prices, Π_h, and then assigns probabilities on those prices to create a forecast, $\tilde{p}_h$. We then ask when the set of possible market prices can be deemed self-fulfilling.

First, no price in Π_h should be ruled out in period 0, or else Π_h would contain infeasible prices; that is, for any $p \in \Pi_h$ there should be a weakly self-fulfilling ETE $(x, \tilde{p}, p)$, i.e., each household's period 0 choice is justified with a weakly self-fulfilling forecast, $\tilde{p}_h$. Secondly, the forecast should not assign any probability to "impossible" prices; that is, it is additionally required that $\tilde{p}_h \in \Pi_h$ occurs with probability one.

[4] Grandmont (1977, 2008), Radner (1982), among others.

Thirdly, we ask that these sets are common across the households to require alignment of set valued forecasts. Thus formally, we define a self-fulfilling price set as follows:

Definition 8 *A subset* Π *of prices (*$\mathbb{R}_+$*) is said to be a self-fulfilling set of price forecasts if for any* $p \in \Pi$ *there is a weakly self fulfilling ETE* $(x, \tilde{p}, p)$ *such that for every* h*,* $\tilde{p}_h \in \Pi$ *holds with probability one.*

As we argued earlier, ETE takes the point of view that efficiency is more primitive than rational expectations about prices. Here we propose a self-fulfilling set of price forecasts as a sort of hybrid between the RE and ETE. That is, efficient allocations can be decentralized with heterogenous forecasts while retaining a form of self-fulfilling property on a consistent set of price expectations.

A self-fulfilling set of price forecasts may also be interpreted as an *equilibrium with ambiguity*, which does not predict a particular configuration of the market price and transactions, but rather offers candidates with a self-fulfilling property, but without probabilistic details of forecasts and random quantities.

A set of PFE prices constitute a self-fulfilling set. Indeed, let Π be such a set. Then by construction, for any $p \in \Pi$, there is a PFE which is an instance of an ETE with a common price forecast $\tilde{p}_h = p$, and trivially $\tilde{p}_h \in \Pi$ holds with probability one for every h.

It is of interest to ask if a self fulfilling set of price forecasts must necessarily be of this form: if so, then the idea of rational expectations can be justified without imposing a common deterministic forecast.

We shall argue that there is a robust class of economies where there exists a self-fulfilling set of price forecasts containing prices other hand PFE prices. We give a general illustration of the idea first, and then examine an example.

4.1 A general construction method

We fix a generic economy and write $\bar{p}$ for a locally unique PFE price of this economy where Proposition 3 applies. Let ϕ_h be the forecast function which maps each ETE price p around $\bar{p}$ to household h's deterministic forecast $\hat{p}_h$. The analysis of CKZ (2018b) further reveals that ϕ_h is a continuously differentiable function locally defined around p

and satisfies $\phi_h(\bar{p}) = \bar{p}$. We let $q(p) = \frac{1}{p}$ and $x(p)$ and $z(p) = e^0 - x^0(p)$ denote the ETE allocation and saving respectively, locally parameterized by p. For convenience we write $\bar{q} = q(\bar{p})$, $\bar{z} = z(\bar{p})$. Finally, let $\hat{q}_h(p) = \frac{1}{\phi_h(p)}$

If every forecast function ϕ_h is a local contraction around $\bar{p}$, we can construct a self-fulfilling set Π which is an interval around $\bar{p}$ containing non PFE prices, provided a regularity condition hold at $\bar{p}$. A formal statement is given below.

Claim 9 *Suppose for each household h, we have (i) $\frac{dk_h^{\bar{z}_h}(\bar{q})}{dq} \neq 0$ and (ii) $-1 < \frac{d\phi_h(\bar{p})}{dp} < 1$ holds. Then there exists an interval Π containing $\bar{p}$ that forms a self-fulfilling set of price forecasts.*

To verify the claim, note that under the hypothesis, for each h, there exists γ_h with $0 < \gamma_h < 1$ and $\nu_h > 0$ such that for any $\nu'_h < \nu_h$, $\phi_h(p) \in [\gamma_h(\bar{p} - \nu'_h), \gamma_h(\bar{p} + \nu'_h)]$ whenever $p \in [\bar{p} - \nu'_h, \bar{p} + \nu'_h]$. Let $\nu = \min\{\nu_h : h = 1, ..., H\}$ and $\gamma = \max\{\gamma_h : h = 1, ..., H\} < 1$. Set $\mathcal{I}_1 := (\bar{p} - \nu, \bar{p} + \nu)$. Furthermore, since $\frac{dk_h^{\bar{z}_h}(\bar{q})}{dq} \neq 0$ and $k_h^{z_h(p)}(\cdot)$ are smooth for every h, there exists an interval $\mathcal{I}_2$ around $\bar{p}$ such that $\frac{dk_h^{z_h(p)}(\hat{q}_h(p))}{dq} \neq 0$ in $\mathcal{I}_2$. In particular, in conjunction with the fact that $k_h^{z_h(p)}(\hat{q}_h(p)) = 0$ at any p around $\bar{p}$, and the fact $k_h^{z_h(p)}(q) = 0$ has a unique solution, $\mathcal{I}_2$ can be chosen so that for every h, $k_h^{z_h(p)}(q(p)) \neq 0$ holds whenever $q(p) \neq \hat{q}_h(p)$ in $\forall p \in \mathcal{I}_2$. Finally, set $\Pi := \mathcal{I}_1 \cap \mathcal{I}_2$.

Then by construction, for any $p \in [\bar{p} - \nu, \bar{p} + \nu]$, $\phi_h(p) \in [\gamma_h(\bar{p} - \nu'_h), \gamma_h(\bar{p} + \nu'_h)] \subset [\gamma(\bar{p} - \nu), \gamma(\bar{p} + \nu)] \subset \Pi$ holds for any h. That is, $\phi_h(p)$ is an interior point of Π. Moreover, by the construction of $\mathcal{I}_1$, for any $p \in \Pi$, $p \neq \bar{p}$ implies that $\phi_h(p) \neq p$ holds for every h. Next, observe that if $\phi_h(p) \neq p$, then $\hat{q}_h(p) \neq q(p)$, which in turn implies that $k^{z_h(p)}(q(p)) \neq 0$ by the construction of $\mathcal{I}_2$. We may now apply Lemma 6 to construct a binary random forecast $\tilde{p}_h$ which takes p and a value close to $\phi_h(p)$ in Π, with which household h demands the same amount as with $\phi_h(p)$. Then we have a weakly self-fulfilling ETE $(x, \tilde{p}, p)$ where each forecast $\tilde{p}_h$ takes a value in Π with probability one. Therefore, Π is a self fulfilling set of price forecasts.

We shall illustrate these ideas using an example below, which also serves as an instance of a robust economy where a non-trivial self-fulfilling set of price forecasts exists.

4.2 An example

The following is the specification of the example which is borrowed from Chatterji, Kajii, Zeng (2018b).

- $H = \{1, 2, 3\}$.
- Endowments: $e_1 = (2 - \varepsilon, \varepsilon), e_2 = (\varepsilon', 2 - \varepsilon')$ and $e_3 = \big(1 + (\varepsilon - \varepsilon'), 1 + (\varepsilon - \varepsilon')\big)$, where $0 < \varepsilon, \varepsilon' < 1$ are given parameters. Note that $\sum_{h=1}^{H} e_h^0 = 3$ and $\sum_{h=1}^{H} e_h^1 = 3$.
- For each $h \in H$, $u_h(x_h^0, x_h^1) = \ln x_h^0 + \ln x_h^1$.

As in the main analysis, we set period-0 price equal to one and the interest rate equal to zero, and write $\hat{p}_1, \hat{p}_2, \hat{p}_3 > 0$ for the deterministic forecasts of the households.

Since the aggregate endowment is the same in both period, a feasible allocation is efficient if and only if it exhibits perfect consumption smoothing. Thus, for all those economies parameterized by ε and ε', the set of Pareto efficient allocations is

$$\mathcal{P} = \left\{ (x_h^0, x_h^1)_{h=1}^{H} \in \mathbb{R}_+^{2\times 3} \middle| (x_h^0, x_h^1) = \alpha_h(1,1), \alpha_h \geq 0 \text{ for all } h \in H, \text{ and } \sum\nolimits_{h=1}^{H} \alpha_h = 1 \right\}.$$

so in particular, the initial endowments are not Pareto efficient. It is readily seen that the unique PFE, occurs at $p^* = 1$, with the allocation $x_h^* = (1, 1)$ for every $h \in H$.

Finding an ETE is cumbersome but it is straightforward enough.[5] If we set the forecast functions as: for p close enough to $p^* = 1$,

$$\phi_1(p) \equiv \hat{p}_1 = \frac{2-\varepsilon}{\varepsilon}\frac{1-p}{1+p} + \frac{2p}{1+p}, \tag{5}$$

$$\phi_2(p) \equiv \hat{p}_2 = \frac{\varepsilon'}{2-\varepsilon'}\frac{1-p}{1+p} + \frac{2p}{1+p}, \text{ and} \tag{6}$$

$$\phi_3(p) \equiv \hat{p}_3 = \frac{1+(\varepsilon-\varepsilon')}{1-(\varepsilon-\varepsilon')}\frac{1-p}{1+p} + \frac{2p}{1+p}, \tag{7}$$

then $\left((\phi_h(p))_{h=1}^3, p\right)$ constitute deterministic ETE forecasts and an ETE price. In particular, forecast functions $(\phi_h)_{h=1}^3$ are exactly the ones discussed in the general analysis in the previous subsection. It can be verified by computation that their derivatives at the PFE price $p^* = 1$ are

$$\phi_h'(1) = \frac{1-\eta_h}{2}, \; h = 1, 2, 3,$$

[5] More details can be found in Chatterji, Kajii, Zeng (2018b).

where $\eta_1 = \frac{2-\varepsilon}{\varepsilon}$, $\eta_2 = \frac{\varepsilon'}{2-\varepsilon'}$ and $\eta_3 = \frac{1+(\varepsilon-\varepsilon')}{1-(\varepsilon-\varepsilon')}$.

It is readily verified that the condition $\frac{dk_h^{z_h^*}(q^*)}{dq} \neq 0$ required in the Claim in the previous subsection holds since the derivative in question is $\frac{e_h^1}{(x_h^{*1})^2}$.

We now examine two special cases of interest.

(i) Assume $\varepsilon = \varepsilon' = 0.7$. It can be readily checked that household 3 does not trade at the PFE and $\phi_3'(1) = 0$. Furthermore, $\phi_1'(1) = -\frac{3}{7}$ and $\phi_2'(1) = \frac{3}{13}$. Therefore, in this economy, we have the configuration $-1 < \phi_h'(p^*) < 1$ for every h. Following the general argument, one can find an interval Π containing $p^* = 1$ that forms a self-fulfilling set of price forecasts. Since all the relevant functions are continuous, the conclusion remains valid for values of ε and ε' that are close to 0.7 and possibly different, i.e., the example is robust. In particular, household 3 not trading at the PFE is not essential in the construction.

(ii) Assume $\varepsilon = \varepsilon' = 0.2$. It can be readily seen that household 3 does not trade at the PFE and $\phi_3'(1) = 0$. Furthermore, $\phi_1'(1) = -4$ and $\phi_2'(1) = \frac{4}{9}$. Therefore this economy does not have the configuration $-1 < \phi_h'(p^*) < 1$ for all $h = 1, ..., H$, and hence the construction method does not work.

5 Concluding Remarks

The idea of a self-fulfilling set of price forecasts is not confined to the simple setup we considered in this paper. Although we do not elaborate on it formally here, it is rather straightforward to state it when there are multiple goods in each period. We believe that an extension of the idea to multiple periods can also be done.

We conjecture that the construction method we have outlined above can be generalized to economies with multiple goods in each period. It would be interesting to investigate conditions under which one may generate self-fulfilling sets of price forecasts in these general set ups.

In the second case in the previous subsection, it appears that the only possible candidate for Π is the singleton set containing the PFE price. Although we have no formal apparatus to verify this non-existence claim, it nonetheless seems that in this economy, the set valued self-fulfilling property is enough to rule out any ETE which does not exhibit perfect foresight. One may further interpret this observation as saying that 'common knowledge' of efficiency and decentralization with price expectations with a self-fulfilling property must necessarily induce perfect foresight. It would be of interest to formalize this observation as a way of providing a foundation to the axiom of rational expectations.

At any rate, the analysis provided in this paper, despite its rudimental nature, offers an avenue for exploring the extent to which the essence of Radner equilibrium can be separated from the rational expectations paradigm. We contend that it gives not only a new foundation based on efficiency and decentralized spot markets, but also an interesting way to extend the concept of Radner equilibrium to allow ambiguity about future prices while retaining a form of self-fulfilling prophecy.

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