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Abstract

A linear-quadratic-Gaussian (LQG) game is an incomplete information game with quadratic payoff functions and Gaussian information structures. It has many applications such as a Cournot game, a Bertrand game, a beauty contest game, and a network game among others. LQG information design is a problem to find an information structure from a given collection of feasible Gaussian information structures that maximizes a quadratic objective function when players follow a Bayes Nash equilibrium. This paper studies LQG information design by formulating it as semidefinite programming, which is a natural generalization of linear programing. Using the formulation, we provide sufficient conditions for optimality and suboptimality of no and full information disclosure. In the case of symmetric LQG games, we characterize the optimal symmetric information structure, and in the case of asymmetric LQG games, we characterize the optimal public information structure, each of which is in a closed-form expression.

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1 Introduction

An equilibrium outcome in an incomplete information game depends not only upon a payoff structure, which consists of payoff functions together with a probability distribution of a payoff state, but also upon an information structure, which maps a payoff state to possibly stochastic signals of players. Information design analyzes the influence of an information structure on equilibrium outcomes, and in particular, characterizes an optimal information structure that induces an equilibrium outcome maximizing the expected value of an objective function of an information designer, who is assumed to be able to choose and commit to the information structure.[1] General approaches to information design are presented by Bergemann and Morris (2013, 2016a,b, 2019), Taneva (2019), and Mathevet et al. (2020). A rapidly growing body of literature have investigated the economic application of information design in areas such as matching markets (Ostrovsky and Schwarz, 2010), voting games (Alonso and Camara, 2016), congestion games (Das et al., 2017), auctions (Bergemann et al., 2017), contests (Zhang and Zhou, 2016), and stress testing (Inostroza and Pavan, 2018), among others.[2]

It would be desirable if we could provide general insights on the connection between optimal information structures and economic environments. Such insights, however, are difficult to obtain by studying design problems that are either too specific or too general. Taneva (2019) provides a complete characterization of the optimal information structures in symmetric 2×2 incomplete information games with symmetric binary states, thus revealing the connection between optimal information structures and the binary symmetric environments. To obtain more insights, we must also consider non-binary environments, but such environments may not be tractable enough if they are too general.

The purpose of this paper is to introduce a tractable class of information design problems which are not too specific. For this purpose, we focus on an incomplete information game such that payoff functions are quadratic in actions and payoff states, and payoff states and players' signals are jointly normally distributed, whose origin goes back to the seminal paper by Radner (1962). Because payoff functions are arbitrary as long as it is quadratic, this class of games encompasses a wide class of interesting economic environments such as Cournot and Bertrand

[1]Kamenica and Gentzkow (2011) phrased the design of optimal information structures as a "Bayesian persuasion" problem between a sender and single receiver. Information design is also referred to as Bayesian persuasion to multiple (interacting) receivers.

[2]See recent survey papers by Bergemann and Morris (2019) and Kamenica (2019).

oligopoly (Vives, 1984, 1999), beauty contests (Morris and Shin, 2002), and network games (Calvó-Armengol et al., 2015), among others. In addition, there exists a unique Bayes Nash equilibrium (BNE) under mild conditions, and it can be calculated as a linear function of signals (Radner, 1962; Ui, 2016a), which ensures high tractability. This class of games is referred to as linear-quadratic Gaussian (LQG) games.

We study information design for LQG games, or *LQG information design* for short. Fix a payoff structure composed of quadratic payoff functions and normally distributed payoff states, which is called a basic game in the literature. An information designer has an objective function that is quadratic in actions and payoff states and can choose and commit to an information structure from a given collection of feasible Gaussian information structures, which determine a joint normal distribution of signals and a payoff state with its marginal distribution being the same as that given by the basic game. LQG information design is a problem to find an optimal information structure that maximizes the expected value of the quadratic objective function over the set of feasible Gaussian information structures.

To analyze the problem, we follow the two-step approach of Bergemann and Morris (2013, 2016b, 2019) and Taneva (2019): the first step identifies the set of inducible equilibrium outcomes under feasible information structures, and the second step identifies the optimal outcomes from the set of inducible outcomes. Especially, we expand on the analysis of Bergemann and Morris (2013), who consider a symmetric basic LQG game with a continuum of players. They identify the set of all symmetric Bayes correlated equilibria (BCE) with normally distributed actions and show that it coincides with the set of all outcomes that can arise in BNE associated with all symmetric Gaussian information structures. Then, focusing on a Cournot game as a special case, they obtain the BCE that maximizes the total expected profit, which is the equilibrium outcome under the information structure that maximizes the total expected profit.

In our first step, we consider a general class of (possibly asymmetric) basic LQG games with an arbitrary number of players, and identify the set of all BCE with normally distributed actions, which coincides with the set of all inducible equilibrium outcomes under Gaussian information structures (Bergemann and Morris, 2013, 2016a). It is well known that the expected values of equilibrium actions do not depend upon the choice of Gaussian information structures in LQG games (Radner, 1962; Ui, 2016a), and thus the expected values of actions are the same for all BCE. Therefore, to represent each BCE, it is enough to determine the covariance matrix of actions and payoff states under the BCE. We show that the set of the covariance matrices representing BCE is the set of positive semidefinite matrices satisfying a linear constraint.

In our second step, we show that, for each BCE, the expected value of a quadratic objective function is represented as a linear function of the covariance matrix representing the BCE, which is a Frobenius inner product of the covariance matrix and the matrix associated with the quadratic form in the objective function. Thus, a BCE maximizes the expected value of the objective function if and only if the covariance matrix representing the BCE is a solution to the problem *to maximize the linear function of a positive semidefinite matrix subject to the linear constraint*. Such an optimization problem is called semidefinite programming (SDP),[3] which is a natural generalization of linear programing.

On the basis of the above discussion, we formulate LQG information design as semidefinite programming not only when every Gaussian information structure is feasible but also when feasibility of information structures is defined in terms of additional linear constraints on the covariance matrix representing BCE. As shown by Taneva (2019), information design with finite sets of actions is reduced to linear programming. This implies that information design with infinite sets of actions, such as LQG information design, can be interpreted as linear programming with infinite number of variables, which is not necessarily tractable. On the other hand, our semidefinite programming formulation of LQG information design enables us not only to numerically obtain the optimal information structures but also to analytically characterize them in some special cases.

As an immediate consequence of the formulation, we obtain simple sufficient conditions for optimality and suboptimality of no information disclosure. If the matrix associated with the quadratic form in a quadratic objective function is negative definite, then no information disclosure is optimal because such an objective function is strictly concave and stochastic actions decrease its expected value. In contrast, if the matrix is positive definite, then no information disclosure is never optimal because such an objective function is strictly convex and stochastic actions increase its expected value.

By focusing on a couple of special cases, we characterize optimal information structures. First, we consider symmetric LQG games assuming that every symmetric Gaussian information structure is feasible. In this class of LQG information design, we can represent a quadratic objective function as a quadratic function of actions, which is independent of payoff states, by means of the first order condition of BCE. This implies that an objective function is written as a

[3]See Vandenberghe and Boyd (1996) and Boyd and Vandenberghe (2004), for example.

linear combination of the variance and the covariance of actions. If the objective function equals the covariance of actions, then full information disclosure is optimal because we can increase the covariance by providing more precise identical signals to players. If the objective function equals the difference between the variance and the covariance of actions, then partial information disclosure is optimal because the difference equals zero under full or no information disclosure. This observation suggests that it is convenient to rewrite an objective function as a linear combination of the covariance and the difference between the variance and the covariance. We characterize the optimal information structure using the ratio of the coefficient of the covariance and that of the difference between the variance and the covariance in the objective function. Ui and Yoshizawa (2015) also consider the same class of symmetric LQG games assuming that each feasible information structure consists of public and private signals and characterize the optimal combination of public and private signals using the same ratio. This paper shows that the insight obtained in Ui and Yoshizawa (2015) is also useful in obtaining the optimal information structures.

We also consider asymmetric LQG games assuming that every public information structure is feasible, where every player receives identical signals. We characterize the optimal public information structure as a closed-form expression. This result also gives a sufficient condition for the optimality of partial information disclosure when all information structures are feasible.

The rest of the paper is organized as follows. Section 2 introduces LQG games and characterizes BNE and BCE. Section 3 formulates LQG information design as semidefinite programming. Section 4 is devoted to symmetric LQG games with symmetric information structures, and Section 5 is devoted to asymmetric LQG games with public information structures.

2 LQG games

2.1 Payoff and information structures

We consider an incomplete information game with quadratic payoff functions and normally distributed payoff states and signals. We call such a game a linear-quadratic Gaussian game, or an LQG game for short.

Let $N = \{1, \dots, n\}$ denote the set of players. Player $i \in N$ chooses a real number $a_i \in A_i \equiv \mathbb{R}$

as his action. The payoff function is

$$u_i(a,\theta) = -q_{ii}a_i^2 - 2\sum_{j\neq i} q_{ij}a_ia_j + 2\theta_ia_i + h_i(a_{-i},\theta), \tag{1}$$

where $a \equiv (a_i)_{i\in N} \in A \equiv \mathbb{R}^N$ is an action profile, $\theta \equiv (\theta_i)_{i\in N} \in \Theta \equiv \mathbb{R}^N$ is a payoff state, q_{ij} is a constant, and $h_i(a_{-i},\theta)$ is an arbitrary function of the opponents' actions $a_{-i} \equiv (a_j)_{j\neq i}$ and a payoff state θ. A payoff state θ is a random vector following a multivariate normal distribution denoted by ψ. We call $G \equiv ((u_i)_{i\in N}, \psi)$ a payoff structure or a basic game, which will be fixed throughout the paper. If $\theta_1 = \cdots = \theta_n$ with probability one, G is called a common value payoff structure. Let $Q = [q_{ij}]_{n\times n}$ denote the matrix consisting of the coefficients in the quadratic terms in (1). We regard vectors such as an action profile a and a payoff state θ as column vectors.

Player $i \in N$ receives a private signal $t_i \in T_i \equiv \mathbb{R}^{m_i}$, which is an m_i-dimensional vector. Let $\pi(t|\theta)$ denote the conditional probability distribution of a signal profile $t \equiv (t_i)_{i\in N}$ given θ, which is referred to as an information structure. An information structure is said to be Gaussian if t and θ are jointly normally distributed, and every information structure in this paper is assumed to be Gaussian. If $t_1 = \cdots = t_n$ with probability one, an information structure is said to be public.

An LQG game consists of a payoff structure (i.e. a basic game) G and an information structure π. For example, consider an LQG game with a symmetric coefficient matrix Q, i.e., $q_{ij} = q_{ji}$ for all $i, j \in N$ in (1), and another LQG game in which players have identical payoff functions given by

$$-a^\top Qa + 2a^\top\theta, \tag{2}$$

where $a^\top$ is the transpose of a. Note that (2) equals (1) if $h_i(a_{-i},\theta) = -\sum_{j\neq i} q_{jj}a_j^2 - 2\sum_{j,k\neq i,\ j\neq k} q_{jk}a_ja_k + \sum_{j\neq i}\theta_ja_j$. An LQG game with identical payoff functions is called an LQG team (Radner, 1962; Ho and Chu, 1972). Clearly, the above two LQG games have the same best response correspondences because the term $h_i(a_{-i},\theta)$ does not have any influence on the player's decision. That is, the best response correspondences of an LQG game with a symmetric coefficient matrix Q coincide with those of an LQG team. Such an LQG game is referred to as a potential game (Monderer and Shapley, 1996) and the identical payoff function (2) of the corresponding LQG team is referred to as a potential function.

We discuss four examples of LQG games, which are also LQG potential games.

Example 1. Firm i produces good i. An action $q_i \in \mathbb{R}$ is the amount of the output and θ_i is the marginal cost. The inverse demand function is $p_i = \alpha - \beta q_i - \gamma \sum_{j \neq i} q_j$. Then, the profit of firm i is

$$\left(\alpha - \beta q_i - \gamma \sum_{j \neq i} q_j\right) q_i - \theta_i q_i.$$

This LQG game is a Cournot game (Vives, 1984, 1999).

Example 2. Firm i produces good i. An action $p_i \in \mathbb{R}$ is the price of the output and θ_i is the marginal cost. The demand function is $q_i = a - bp_i + c \sum_{j \neq i} p_j$. Then, the profit of firm i is

$$\left(a - bp_i + c \sum_{j \neq i} p_j\right) p_i - \theta_i \left(a - bp_i + c \sum_{j \neq i} p_j\right).$$

This LQG game is a Bertrand game (Vives, 1984, 1999).

Example 3. Let a coefficient matrix Q be given by

$$q_{ij} = q_{ji} = \begin{cases} 1 & \text{if } i = j, \\ 0 \text{ or } \alpha & \text{if } i \neq j, \end{cases}$$

where $\alpha \neq 0$ is a constant. The matrix Q represents interaction of players located on a graph with the set of vertices N and the set of edges $E = \{(i, j) \in N^2 : i \neq j,\ q_{ij} = \alpha\}$. When θ is constant, this is a network game studied by Ballester et al. (2006) and Bramoullé et al. (2014). Calvó-Armengol et al. (2015) use an LQG network game to analyze endogenous communication in organizations.

Example 4. Consider an LQG game with a common value payoff structure and a coefficient matrix Q with

$$q_{ij} = \begin{cases} 1/(1-r) & \text{if } i = j, \\ r/(1-r) \times 1/(n-1) & \text{if } i \neq j, \end{cases}$$

where $r \in (0, 1)$. Then, the best response equals the conditional expected value of the weighted average of the arithmetic mean of the opponents' actions and the common payoff state,[4] i.e.,

$$\sigma_i(t_i) = r \sum_{j \neq i} E_\pi[\sigma_j | t_i]/(n-1) + (1-r) E_\pi[\theta_0 | t_i].$$

This LQG game is a beauty contest game studied by Morris and Shin (2002).

[4]See (5) in the next section

2.2 Bayes Nash equilibria and Bayes correlated equilibria

Player i's strategy $\sigma_i : T_i \to A_i$ assigns an action $\sigma_i(t_i) \in A_i$ to each realization of a private signal $t_i \in T_i$. A strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a Bayes Nash equilibrium (BNE) under an information structure π if, for all $a'_i \in A_i$, $t_i \in T_i$, and $i \in N$, it holds that

$$E_\pi[u_i((\sigma_i(t_i), \sigma_{-i}), \theta)|t_i] \geq E_\pi[u_i((a'_i, \sigma_{-i}), \theta)|t_i], \tag{3}$$

where $\sigma_{-i} = \sigma_{-i}(t_{-i}) = (\sigma_j(t_j))_{j \neq i}$ and $E_\pi[\cdot|t_i]$ is the conditional expectation operator given $t_i \in T_i$ with respect to π and ψ. The first-order condition for a BNE is

$$E_\pi\left[\frac{\partial}{\partial a_i} u_i(\sigma(t), \theta)\,\Big|\, t_i\right] = -2q_{ii}\sigma_i(t_i) - 2\sum_{j \neq i} q_{ij} E_\pi[\sigma_j|t_i] + 2E_\pi[\theta_i|t_i] = 0 \tag{4}$$

for all $t_i \in T_i$ and $i \in N$, which is reduced to

$$\sigma_i(t_i) = \sum_{j \neq i} q_{ij} E_\pi[\sigma_j|t_i]/q_{ii} + E_\pi[\theta_i|t_i]/q_{ii}. \tag{5}$$

Thus, the best response is the conditional expected value of the weighted sum of the opponents' actions and the payoff state.

Now assume that an information structure π is public with $t_0 = t_1 = \cdots = t_n$, which we call a public signal under π. Then, (4) is reduced to

$$\sum_{j \in N} q_{ij}\sigma_j(t_0) = E_\pi[\theta_i|t_0].$$

Thus, if Q is nonsingular, then the equilibrium action profile is $(\sigma_i(t_0))_{i \in N} = Q^{-1}E_\pi[\theta|t_0]$ when t_0 is realized. For example, when θ is common knowledge, the equilibrium action profile is $Q^{-1}\theta$; when players do not receive any information (i.e., do not update their prior), the equilibrium action profile is $\bar{a} \equiv Q^{-1}\bar{\theta}$, where $\bar{\theta} \equiv E[\theta]$.

Even if an information structure is not public, we can obtain a unique BNE as linear functions of private signals (Radner, 1962; Ui, 2016a).[5]

Proposition 1. *Suppose that $Q + Q^\top$ and* $\mathrm{var}(t_i)$ *are positive definite for each $i \in N$. Then, an LQG game has a unique BNE given by*

$$\sigma_i(t_i) = \bar{a}_i + b_i^\top(t_i - E_\pi[t_i]) \text{ for } i \in N, \tag{6}$$

[5] Radner (1962) was the first to introduce an LQG team and establishes Proposition 1. As pointed out by Ui (2009), we can directly use Radner's result to obtain a unique BNE in an LQG potential game, which corresponds to Proposition 1 with a symmetric coefficient matrix Q. Ui (2016a) shows that the symmetry assumption is not necessary.

where $(\bar{a}_i)_{i\in N} = Q^{-1}\theta$ *and* $b_1, \dots, b_n$ *are determined by the following system of linear equations:*

$$\sum_{j\in N} q_{ij}\mathrm{cov}(t_i, t_j)b_j = \mathrm{cov}(t_i, \theta_i) \textit{ for } i \in N. \tag{7}$$

In the rest of this paper, we assume that $Q + Q^\top$ and $\mathrm{var}(t_i)$ are positive definite for each $i \in N$ to ensure the existence and uniqueness of a BNE. Positive definiteness of $\mathrm{var}(t_i)$ implies that any component of t_i is not informationally redundant. Let Π^* denote the set of all Gaussian information structures such that $\mathrm{var}(t_i)$ are positive definite for each $i \in N$.

Imagine that players follow a strategy profile σ under an information structure π. The outcome of this situation is described by the conditional probability distribution of a given θ because the marginal probability distribution of θ is fixed. We denote this conditional distribution by $\rho(a|\theta)$ and call it an action distribution of σ under π, which is given by $\rho(a|\theta) = \sum_{t:\sigma(t)=a} \pi(t|\theta)$. In particular, when σ is a BNE under π, we call ρ an equilibrium action distribution under π. Every equilibrium action distribution ρ satisfies the following condition:

$$E_\rho[u_i((a_i, a_{-i}), \theta)|a_i] \geq E_\rho[u_i((a_i', a_{-i}), \theta)|a_i] \text{ for all } a_i, a_i' \in A_i \text{ and } i \in N, \tag{8}$$

where $E_\rho[\cdot|a_i]$ is the conditional expectation operator given a_i with respect to ρ and ψ. This is because, by (3), it holds that

$$\begin{aligned} E_\rho[u_i((a_i, a_{-i}), \theta)|a_i] &= E_\pi[u_i((a_i, \sigma_{-i}), \theta)|\sigma_i(t_i) = a_i] \\ &\geq E_\pi[u_i((a_i', \sigma_{-i}), \theta)|\sigma_i(t_i) = a_i] = E_\rho[u_i((a_i', a_{-i}), \theta)|a_i]. \end{aligned}$$

We say that an action distribution ρ is a Bayes correlated equilibrium (BCE) if it satisfies (8). Thus, an equilibrium action distribution is a BCE because it satisfies (8). Moreover, (8) implies that a BCE is an equilibrium action distribution under π satisfying the following conditions:

1. $T_i = A_i$ for all $i \in N$; that is, the set of player i's signals coincides with the set of player i's actions.

2. For a BNE σ under π, $\sigma_i(a_i) = a_i$ for all $a_i \in T_i = A_i$ and $i \in N$; that is, an equilibrium action distribution is given by $\pi(a|\theta)$.

The above discussion is summarized in the following proposition due to Bergemann and Morris (2013).

Proposition 2. *Under any information structure, an equilibrium action distribution is a BCE. For any BCE, there exists an information structure under which the equilibrium action distribution coincides with the BCE.*

Using Propositions 1 and 2, we can obtain a necessary and sufficient condition for an action distribution to be a BCE where an action profile and a payoff state are jointly normally distributed

Proposition 3. *An action distribution ρ is a BCE under which (a, θ) is normally distributed if and only if the following condition is satisfied.*

$$E_\rho[a] = \bar{a}, \tag{9}$$

$$\sum_{j \in N} q_{ij} \mathrm{cov}(a_i, a_j) = \mathrm{cov}(a_i, \theta_i). \tag{10}$$

Proof. A necessary and sufficient condition for a BCE is (6) and (7) with $b_i = 1$ and $\bar{a}_i = E_\pi[t_i]$ by Proposition 1, which establishes this proposition. □

Bergemann and Morris (2013) obtain Proposition 3 in the case of symmetric LQG games with common value payoff structures.

3 LQG information design

3.1 A general formulation

Fix a payoff structure (i.e. a basic game) $G = ((u_i)_{i \in N}, \psi)$ such that $Q + Q^\top$ is positive definite. We consider an information designer who chooses an information structure π from a set of feasible information structures $\Pi \subseteq \Pi^*$ to maximize the expected value of a quadratic objective function $v(a, \theta)$. The designer can make a commitment to provide information according to the following timeline, which is a standard assumption in the literature.

1. The designer chooses $\pi \in \Pi$ and informs all players of π.
2. When θ is realized, $t_1, \ldots, t_n$ are drawn according to $\pi(t|\theta)$.
3. Players follow the unique BNE under π.

Note that we have focused on a payoff structure such that any information structure induces a unique equilibrium.[6]

[6]See Mathevet et al. (2020) for detailed discussions on information design with multiple equilibria.

Without loss of generality, we omit linear terms in a quadratic objective function $v(a, \theta)$ because the expected value of (a, θ) is a constant vector $(\bar{a}, \bar{\theta})$ in a BNE under any information structure. Thus, we use a quadratic form to represent $v(a, \theta)$. Let $\mathcal{S}^k$ and $\mathcal{S}^k_+$ denote the set of all $k \times k$ symmetric matrices and the set of all $k \times k$ positive semidefinite symmetric matrices, respectively. Then, a quadratic objective function is given by

$$v(a, \theta) = [a^\top, \theta^\top] V \begin{bmatrix} a \\ \theta \end{bmatrix} = \mathrm{tr}\left(V \begin{bmatrix} aa^\top a\theta^\top \\ \theta a^\top \theta\theta^\top \end{bmatrix} \right) = \mathrm{tr}\left(V_{11} aa^\top\right) + 2\mathrm{tr}\left(V_{12}\theta a^\top\right) + \mathrm{tr}\left(V_{22}\theta\theta^\top\right), \tag{11}$$

where

$$V = [v_{ij}]_{2n \times 2n} = \begin{bmatrix} [v_{ij}]_{n\times n} & [v_{i,n+j}]_{n\times n} \\ [v_{n+i,j}]_{n\times n} & [v_{n+i,n+j}]_{n\times n} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \in \mathcal{S}^{2n}.$$

We can assume that $V_{22} = O$, i.e., $v_{n+i,n+j} = 0$ for all $i, j \in N$, because the expected value of $\mathrm{tr}\,(V_{22}\theta\theta^\top)$ is a constant determined by the payoff structure G.

An LQG information design problem is the problem to find an information structure that maximizes the expected value of the objective function over the set of feasible information structures:

$$\max_{\pi \in \Pi} E_\pi[v(\sigma, \theta)], \tag{12}$$

where E_π is the expectation operator with respect to π and σ is the unique BNE under π. Using the equilibrium action distribution ρ under π, we can replace $E_\pi[v(\sigma, \theta)]$ in (12) with $E_\rho[v(a, \theta)]$, where E_ρ is the expectation operator with respect to ρ. Thus, (12) is equivalent to

$$\max_{\rho \in C(\Pi)} E_\rho[v(a, \theta)], \tag{13}$$

where

$$C(\Pi) = \{\rho : \rho \text{ is the equilibrium action distribution under } \pi \in \Pi\}.$$

Using the representation (11), we can rewrite the objective function in (13) as

$$E_\rho[v(a, \omega)] = \mathrm{tr}\left(V \begin{bmatrix} \mathrm{var}(a) & \mathrm{cov}(a, \theta) \\ \mathrm{cov}(\theta, a) & \mathrm{var}(\theta) \end{bmatrix} \right) + \mathrm{tr}\left(V \begin{bmatrix} \bar{a}\bar{a}^\top \bar{a}\bar{\theta}^\top \\ \bar{\theta}\bar{a}^\top \bar{\theta}\bar{\theta}^\top \end{bmatrix} \right) = V \bullet X + \text{const.}, \tag{14}$$

where

$$X = [x_{ij}]_{2n \times 2n} = \begin{bmatrix} \mathrm{var}(a) & \mathrm{cov}(a, \theta) \\ \mathrm{cov}(\theta, a) & \mathrm{var}(\theta) \end{bmatrix} \in \mathcal{S}^{2n}_+ \text{ and } V \bullet X = \sum_{i=1}^{2n} \sum_{j=1}^{2n} v_{ij} x_{ij}.$$

Thus, (13) is equivalent to

$$\max_{X \in \mathcal{X}(\Pi)} V \bullet X, \tag{15}$$

where

$$\mathcal{X}(\Pi) = \{X \in \mathcal{S}_+^{2n} : X \text{ is the covariance matrix of } (a, \theta) \text{ under } \rho \in C(\Pi)\}.$$

Note that there is a one-to-one correspondence between the solution to (13) and that to (15) because $\mathcal{X}(\Pi)$ is the collection of the covariance matrices of (a, θ) under the equilibrium action distributions.

For example, when players receive no signals, they follow $Q^{-1}\bar{\theta}$ in the equilibrium, and thus

$$X = \begin{bmatrix} O & O \\ O & \mathrm{var}(\theta) \end{bmatrix} \text{ and } V \bullet X = O.$$

This case is referred to as no information disclosure. When θ is common knowledge, players follow $Q^{-1}\theta$ in the equilibrium, and thus

$$X = \begin{bmatrix} Q^{-1}\mathrm{var}(\theta)(Q^{-1})^\top & Q^{-1}\mathrm{var}(\theta) \\ \mathrm{var}(\theta)(Q^{-1})^\top & \mathrm{var}(\theta) \end{bmatrix} \text{ and } V \bullet X = V_Q \bullet \mathrm{var}(\theta),$$

where

$$V_Q \equiv (Q^{-1})^\top (V_{11} + V_{12}Q + Q^\top V_{21}) Q^{-1}.$$

This case is referred to as full information disclosure. The other cases are referred to as partial information disclosure.

As an immediate consequence of (15), we can obtain the following simple sufficient conditions for optimality and suboptimality of no information disclosure.

Proposition 4. *If V is negative semidefinite, then no information disclosure is optimal in* Π^*. *If V is positive definite, then no information disclosure is not optimal in* Π^*.

Proof. It is well known that if V is negative semidefinite, then $V \bullet X \leq 0$ for any $X \in \mathcal{S}_+^{2n}$, which implies that no information disclosure is optimal. If V is positive definite, then

$$V \bullet X = E_\rho \left[([a^\top - \bar{a}^\top, \theta^\top - \bar{\theta}^\top]) V \left(\begin{bmatrix} a - \bar{a} \\ \theta - \bar{\theta} \end{bmatrix} \right) \right] > 0$$

under partial or full information disclosure, so no information disclosure is not optimal. □

3.2 Semidefinite programming formulation

We consider a special case of (15) in which all information structures are feasible:

$$\max_{X \in \mathcal{X}(\Pi^*)} V \bullet X. \tag{16}$$

The constraint $X \in \mathcal{X}(\Pi^*)$ consists of the following three conditions by Proposition 3.

(i) $\mathrm{var}(\theta) = [\mathrm{cov}(\theta_i, \theta_j)]_{n\times n} = [x_{n+i,n+j}]_{n\times n}$ is the covariance matrix of θ given by a payoff structure G. This condition is rewritten as

$$S_{kl} \bullet X = \mathrm{cov}(\theta_k, \theta_l) \text{ for all } k, l \in \{1, \dots, n\} \text{ with } k \le l, \tag{17}$$

where $S_{kl} = [s_{kl,ij}]_{2n\times 2n} \in \mathcal{S}^{2n}$ is given by, for $i, j \in \{1, \dots, 2n\}$ with $i \le j$,

$$s_{kl,ij} = \begin{cases} 1/2 & \text{if } k < l,\ i = n+k,\ j = n+l, \\ 1 & \text{if } k = l,\ i = n+k,\ j = n+l, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $\sum_{j\in N} q_{ij}\mathrm{cov}(a_i, a_j) = \mathrm{cov}(a_i, \theta_i)$; that is, $\sum_{j\in N} q_{ij} x_{ij} = x_{i,n+i}$ for all $i \in N$. This condition is rewritten as

$$R_k \bullet X = 0 \text{ for all } k \in \{1, \dots, n\}, \tag{18}$$

where $R_k = [r_{k,ij}]_{2n\times 2n} \in \mathcal{S}^{2n}$ is given by

$$r_{k,ij} = \begin{cases} q_{kk} & \text{if } i = j = k, \\ q_{kj}/2 & \text{if } i = k,\ 1 \le j \le n,\ j \ne k, \\ -1/2 & \text{if } i = k,\ j = n+k, \\ q_{ki}/2 & \text{if } j = k,\ 1 \le i \le n,\ i \ne k, \\ -1/2 & \text{if } j = k,\ i = n+k, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) X is a positive semidefinite matrix, i.e., $X \in \mathcal{S}^{2n}_+$.

Thus, (16) is reduced to a problem to *maximize a linear function of a positive semidefinite matrix X subject to linear constraints* (i) and (ii):

$$\begin{aligned} \max V \bullet X \text{ s.t. } & R_k \bullet X = 0 \text{ for all } k \in \{1, \dots, n\}, \\ & S_{kl} \bullet X = \mathrm{cov}(\theta_k, \theta_l) \text{ for all } k, l \in \{1, \dots, n\} \text{ with } k \le l, \\ & X \in \mathcal{S}^{2n}_+. \end{aligned} \tag{19}$$

Such a problem is called a semidefinite programming (SDP) problem.[7]

We have shown that (15) is reduced to a SDP problem when $\Pi = \Pi^*$. Nonetheless, even when $\Pi \subsetneq \Pi^*$, if $\mathcal{X}(\Pi) \subsetneq \mathcal{X}(\Pi^*)$ is given by some linear constraints on $X \in \mathcal{X}(\Pi^*)$, i.e., there exist $M_1, \ldots, M_K \in \mathcal{S}^{2n}$ and $m_1, \ldots, m_K \in \mathbb{R}$ such that

$$\mathcal{X}(\Pi) = \{X \in \mathcal{X}(\Pi^*) : M_k \bullet X = m_k \text{ for all } k \in \{1, \ldots, K\}\},$$

then (15) is reduced to a SDP problem to maximize a linear function of a positive semidefinite matrix X subject to linear constraints (i), (ii), and $M_k \bullet X = m_k$ for all $k \in \{1, \ldots, K\}$.

The KKT condition for a SDP problem is well known. The next proposition gives that for (19).

Proposition 5. *If $\bar{X} \in \mathcal{S}_+^{2n}$ is a solution to* (19)*, then there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\mu} \in \mathbb{R}^{n(n+1)/2}$, $\bar{\Xi} \in \mathcal{S}_+^{2n}$ satisfying the following condition.*

$$-V - \sum_{k=1}^{n} \bar{\lambda}_k R_k - \sum_{k=1}^{n} \sum_{l=l}^{k} \bar{\mu}_{kl} S_{kl} = \bar{\Xi},$$

$$R_k \bullet \bar{X} = 0 \textit{ for all } k \in \{1, \ldots, n\},$$

$$S_{kl} \bullet \bar{X} = \mathrm{cov}(\theta_k, \theta_l) \textit{ for all } k, l \in \{1, \ldots, n\} \textit{ with } k \leq l,$$

$$\bar{X} \bullet \bar{\Xi} = 0.$$

Conversely, if there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\mu} \in \mathbb{R}^{n(n+1)/2}$, $\bar{\Xi}, \bar{X} \in \mathcal{S}_+^{2n}$ satisfying the above condition, then $\bar{X}$ is a solution to (19)*.*

On the basis of the SDP formulation of LQG information design, we can numerically obtain optimal information structures using SDP solvers, and in some cases, we can analytically obtain them, as will be discussed in the subsequent sections.

Before closing this subsection, we focus on the case of common value payoff structures with $\theta_0 = \theta_1 = \cdots = \theta_n$, which will be discussed in Section 4. To represent an objective function, it is enough to consider the covariance matrix of (a, θ_0) denoted by

$$X' = [x_{ij}]_{(n+1)\times(n+1)} \equiv \begin{bmatrix} \mathrm{var}(a) & \mathrm{cov}(a, \theta_0) \\ \mathrm{cov}(\theta_0, a) & \mathrm{var}(\theta_0) \end{bmatrix} \in \mathcal{S}_+^{n+1}.$$

[7]See Vandenberghe and Boyd (1996) and Boyd and Vandenberghe (2004), for example.

Thus, by appropriately choosing $V' \in \mathcal{S}^{n+1}$, we can rewrite (19) as

$$\begin{aligned} \max V' \bullet X' \text{ s.t. } & R_k \bullet X' = 0 \text{ for all } k \in \{1, \dots, n\}, \\ & S \bullet X' = \mathrm{var}(\theta_0), \\ & X \in \mathcal{S}_+^{n+1}, \end{aligned} \tag{20}$$

where $R_k = [r_{k,ij}]_{(n+1)\times(n+1)} \in \mathcal{S}^{n+1}$ and $S = [s_{ij}]_{(n+1)\times(n+1)} \in \mathcal{S}^{n+1}$ are given by

$$r_{k,ij} = \begin{cases} q_{kk} & \text{if } i = j = k, \\ q_{kj}/2 & \text{if } i = k,\ 1 \le j \le n,\ j \ne k, \\ -1/2 & \text{if } i = k,\ j = n+1, \\ q_{ki}/2 & \text{if } j = k,\ 1 \le i \le n,\ i \ne k, \\ -1/2 & \text{if } j = k,\ i = n+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{ij} = \begin{cases} 1 & \text{if } i = n+1,\ j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Public information structures

Let $\Pi^p \subseteq \Pi^*$ denote the collection of all public information structures. We discuss a special case of (15) when $\Pi = \Pi^p$. As an immediate consequence of the formulation, we provide sufficient conditions for optimality of no or full information disclosure in the set of public information structures and sufficient conditions for suboptimality of no or full information disclosure in the set of all information structures.

Recall that, under a public information structure $\pi \in \Pi^p$, the equilibrium action profile is $Q^{-1}E_\pi[\theta|t_0]$, where t_0 is a public signal. When players follow $Q^{-1}E_\pi[\theta|t_0]$, the covariance matrix of (a, θ) is given by

$$X = \begin{bmatrix} Q^{-1}\mathrm{var}(E_\pi[\theta|t_0])(Q^{-1})^\top & Q^{-1}\mathrm{var}(E_\pi[\theta|t_0]) \\ \mathrm{var}(E_\pi[\theta|t_0])(Q^{-1})^\top & \mathrm{var}(\theta) \end{bmatrix},$$

and thus we have

$$V \bullet X = V_Q \bullet \mathrm{var}(E_\pi[\theta|t_0]).$$

This implies that we can obtain the optimal public information structure by solving

$$\max_{Z \in \mathcal{Z}} V_Q \bullet Z, \tag{21}$$

where $\mathcal{Z} = \{Z : Z = \mathrm{var}(E_\pi[\theta|t_0]),\ \pi \in \Pi^p\}\}$. Note that $\mathcal{Z}$ is the set of all covariance matrices of the conditional expected value of θ under $\pi \in \Pi^p$, which is characterized as follows.

Lemma 1. *Let D be an $n \times k$ matrix of rank k such that $\mathrm{var}(\theta) = DD^\top$ and k is the rank of $\mathrm{var}(\theta)$, which is known to exist. Then, it holds that $\mathcal{Z} = \{Z : Z = DSD^\top,\ S \in \mathcal{S}_+^k,\ I - S \in \mathcal{S}_+^k\}$.*

Proof. There exist k random variables $\xi_1, \ldots, \xi_k \in \mathbb{R}$ that are independently and identically distributed according to the standard normal distribution such that $\theta = D\xi + \bar{\theta}$. Thus,

$$\mathrm{var}(E_\pi[\theta|t_0]) = \mathrm{var}(DE_\pi[\xi|t_0]) = D\mathrm{var}(E_\pi[\xi|t_0])D^\top.$$

Because $S = E_\pi[\xi|t_0] \in \mathcal{S}_+^k$ can be arbitrary as long as $S \in \mathcal{S}_+^k$ and $I - S \in \mathcal{S}_+^k$, this lemma holds. □

By this lemma, for each $Z \in \mathcal{Z}$, there exists $S \in \mathcal{S}_+^k$ with $Z = DSD^\top$, and it holds that

$$V_Q \bullet Z = V_Q \bullet DSD^\top = \mathrm{tr}\,(V_Q DSD^\top) = \mathrm{tr}\,(D^\top V_Q DS) = D^\top V_Q D \bullet S.$$

Thus, (21) is reduced to

$$\max_S\ D^\top V_Q D \bullet S \text{ s.t. } S \in \mathcal{S}_+^k \text{ and } I - S \in \mathcal{S}_+^k. \tag{22}$$

If $D^\top V_Q D = O$, then every information structure is optimal, so we assume that $D^\top V_Q D \neq O$. We will solve (22) in Section 5, but we discuss a couple of immediate consequences of (22) here. The following proposition provides a sufficient condition for optimality of no information disclosure in Π^p.

Proposition 6. *Suppose that $D^\top V_Q D \neq O$ is negative semidefinite. Then, no information disclosure is optimal in Π^p, and full information disclosure is not optimal in Π^p.*

Proof. The matrix $D^\top V_Q D$ is factored as $D^\top V_Q D = U\Lambda U^\top$, where U is orthogonal and Λ is diagonal. Thus,

$$D^\top V_Q D \bullet S = \mathrm{tr}[D^\top V_Q DS] = \mathrm{tr}[U\Lambda U^\top S] = \mathrm{tr}[\Lambda U^\top SU] = \sum_{l=1}^k \lambda_l \gamma_l, \tag{23}$$

where λ_l and γ_l are the l-th diagonal elements of Λ and $U^\top SU$, respectively. Note that $0 \leq \gamma_l \leq 1$ for all l because $U^\top SU$ and $U^\top(I - S)U = I - U^\top SU$ are positive semidefinite. Because $D^T V_Q D \neq O$ is negative semidefinite, $\lambda_l \leq 0$ for all l and $\lambda_l < 0$ for at least one l. Thus, $\gamma_l = 0$ for all l is optimal, and $\gamma_l = 1$ for all l is not optimal. The former case corresponds to no information disclosure $S = 0$ and the latter case corresponds to full information disclosure $S = I$. □

Similarly, the following proposition provides a sufficient condition for optimality of full information disclosure in Π^p.

Proposition 7. *Suppose that $D^\top V_Q D \neq O$ is positive semidefinite. Then, full information disclosure is optimal in Π^p, and no information disclosure is not optimal in Π^p.*

Proof. Because $D^T V_Q D \neq O$ is positive semidefinite, $\lambda_l \geq 0$ for all l and $\lambda_l > 0$ for at least one l in (23). Thus, $\gamma_l = 1$ for all l is optimal, and $\gamma_l = 0$ for all l is not optimal. □

For example, consider a common value payoff structure. Because $\theta_0 = \theta_1 = \cdots = \theta_n$, we have $D = \mathrm{var}(\theta_0)^{1/2}\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^n$ is the n-dimensional column vector with all entries one. Thus, by Propositions 6 and 7, full information disclosure is optimal in Π^p if $\mathbf{1}^\top V_Q \mathbf{1} > 0$, and no information disclosure is optimal in Π^p if $\mathbf{1}^\top V_Q \mathbf{1} < 0$.

Propositions 6 and 7 have the following implication for suboptimality of no or full information disclosure in Π^*.

Corollary 8. *If $D^\top V_Q D \neq O$ is negative semidefinite, then full information disclosure is not optimal in Π^*. If $D^\top V_Q D \neq O$ is positive semidefinite, then no information disclosure is not optimal in Π^*.*

In Section 5, we characterize optimal public information structures when $D^\top V_Q D$ is neither positive semidefinite nor negative semidefinite and show that partial information disclosure is optimal in Π^p, which also implies that partial information disclosure is optimal in Π^* as well.

3.4 Some characterization when $v_{i,n+j} = 0$ for $i \neq j$

We consider a special case in which the correlation between player i's action a_i and player j's payoff state θ_j has no influence on the objective function; that is, $v_{i,n+j} = 0$ for all $i, j \in N$ with $i \neq j$. The following two cases are typical.

- Player i's payoff function $u_i(a, \theta)$ is independent of θ_j and an objective function $v(a, \theta)$ is the sum of all players' payoff functions, i.e., $v(a, \theta) = \sum_{i \in N} u_i(a, \theta)$.
- A payoff structure is of common value (i.e., $\theta_1 = \cdots = \theta_n$), where we can replace $a_i\theta_j$ with $a_i\theta_i$, thus making the coefficient of $a_i\theta_j$ zero.

In this case, we can obtain a simple representation of $V \bullet X$ in (15). Although $V \bullet X$ is a linear function of the covariance matrix of an action profile a and a payoff state θ, we can replace it with a linear function of the covariance matrix of an action profile alone.

Lemma 2. *Let* $V = [v_{ij}]_{2n\times 2n} \in \mathcal{S}_{2n}$ *and* $W = [w_{ij}]_{n\times n} \in \mathcal{S}_n$ *be such that* $v_{i,n+j} = 0$ *for all* $i, j \in N$ *with* $i \neq j$ *and* $w_{ij} = v_{ij} + v_{i,n+i}q_{ij} + v_{j,n+j}q_{ji}$ *for all* $i, j \in N$*. If*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{var}(a) & \mathrm{cov}(a,\theta) \\ \mathrm{cov}(\theta,a) & \mathrm{var}(\theta) \end{bmatrix} \in \mathcal{X}(\Pi),$$

then

$$V \bullet X = W \bullet X_{11}.$$

Proof. Because $v_{i,n+j} = 0$ for all $i, j \in N$ with $i \neq j$ and $v_{n+i,n+j} = 0$ for all $i, j \in N$, we have

$$V \bullet X = \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}\mathrm{cov}(a_i, a_j) + 2\sum_{i=1}^{n} v_{i,n+i}\mathrm{cov}(a_i, \theta_i).$$

By plugging (10) into the above,

$$V \bullet X = \sum_{i=1}^{n}\sum_{j=1}^{n}(v_{ij} + 2v_{i,n+i}q_{ij})\mathrm{cov}(a_i, a_j) = W \bullet X_{11},$$

which establishes the lemma. □

By Lemma 2, we can reformulate (15) as

$$\max_{X_{11}:\, X\in\mathcal{X}(\Pi)} W \bullet X_{11}. \tag{24}$$

Using (24), we provide simple sufficient conditions for optimality and suboptimality of no information disclosure, which is analogous to Proposition 4.

Proposition 9. *Suppose that no information disclosure is feasible. If* W *is negative semidefinite, then it is optimal. If* W *is positive definite, then it is not optimal.*

Proof. The proof is essentially the same as that of Proposition 4, so it is omitted. □

On the other hand, if Q is symmetric and W equals a constant times Q, then full information disclosure is optimal.

Proposition 10. *Suppose that full information disclosure is feasible. If* Q *is symmetric and there exists* $c > 0$ *such that* $W = cQ$*, then it is optimal.*

Proof. Consider an LQG team with a payoff function (2) and let $v(a,\theta) = \sum_{i\in N} u_i(a,\theta)/n$. Because every player's payoff function coincides with the objective function, full information disclosure is optimal. In this case, $V = [v_{ij}]_{2n\times 2n}$ is given by

$$v_{ij} = \begin{cases} q_{ij} & \text{if } 1 \leq i, j \leq n, \\ 1 & \text{if } 1 \leq j \leq n,\ i = n + j \text{ or } 1 \leq i \leq n,\ j = n + i, \\ 0 & \text{otherwise,} \end{cases}$$

which satisfies the condition in Lemma 2. Thus, by Lemma 2, it holds that $V \bullet X = W \bullet X_{11}$, where $W = 3Q$. This implies that full information disclosure is optimal if $W = cQ$ for any $c > 0$. □

As applications of Propositions 9 and 10, we compare two types of public goods games with different payoff structures.

Example 5. Player i's contribution is a_i, the marginal cost is θ_i, and the production of a public good is $-a^\top Ma + 2\gamma^\top a$, where $M \in \mathcal{S}^n_+$ is a positive definite matrix and $\gamma \in \mathbb{R}^n$ is a constant vector. Then, the payoff function is $(-a^\top Ma + 2\gamma^\top a) - \theta_i a_i$. Let $v(a, \theta) = \sum_{i\in N} u_i(a, \theta)$; that is, the objective function is the total payoff. Then, the condition in Lemma 2 is satisfied, and W is shown to be negative definite. Therefore, no information disclosure is optimal by Proposition 9. This is because the production is independent of θ and concave in a, so we can increase the expected total payoff by making a constant. Teoh (1997) shows similar optimality of no information disclosure in a public goods game, which is not an LQG game.

Example 6. Player i's contribution is a_i, the marginal cost is a constant γ_i, and the production of a public good is $-a^\top Ma + 2\theta^\top a$, where $M \in \mathcal{S}^n_+$ is a positive definite matrix. Then, the payoff function is $(-a^\top Ma + 2\theta^\top a) - \gamma_i a_i$. Let $v(a, \theta) = \sum_{i\in N} u_i(a, \theta)$; that is, the objective function is the total payoff. Then, the condition in Lemma 2 is satisfied, and W is shown to be a constant times Q. Therefore, full information disclosure is optimal by Proposition 10. This is because the production level depends upon θ, so we can increase the expected total payoff by allowing a to adjust to θ.

4 Symmetric common value payoff structures

In this section, we solve an LQG information design problem by focusing on symmetric common value payoff structures and symmetric information structures. A payoff structure is said to be symmetric if q_{ii} and q_{ij} are constant across $i, j \in N$ with $i \neq j$. By normalizing $q_{ii} = 1$ for all $i \in N$, we obtain the following payoff function:

$$u_i(a, \theta) = -a_i^2 + 2\alpha a_i \frac{\sum_{j \neq i} a_j}{n-1} + 2\theta_0 a_0 + h_i(a_{-i}, \theta_0).$$

This game exhibits strategic complementarities if $\alpha > 0$ and strategic substitutabilities if $\alpha < 0$. Note that a coefficient matrix Q is given by

$$Q = \begin{bmatrix} 1 & q & \cdots & q \\ q & 1 & \cdots & q \\ \vdots & \vdots & \ddots & \\ q & q & \cdots & 1 \end{bmatrix} \in \mathcal{S}^n \tag{25}$$

with $q = -\alpha/(n-1)$. Because the k-th leading principal minor is $(1 + (k-1)q)(1-q)^{k-1}$, Q is positive definite if and only if $-(n-1) < \alpha < 1$, which is assumed throughout this section.

Because the payoff structure is of common value, we solve the problem (20) restricting attention to symmetric information structures. An information structure $\pi \in \Pi^*$ is said to be symmetric if T_i, $\mathrm{var}(t_i)$, and $\mathrm{cov}(t_i, t_j)$ are constant across $i, j \in N$ with $i \neq j$. In the equilibrium, we can write $\sigma_a^2 = \mathrm{var}(a_i)$, $\rho_a \sigma_a^2 = \mathrm{cov}(a_i, a_j)$, $\rho_{a\theta}\sigma_a\sigma_\theta = \mathrm{cov}(a_i, \theta_0)$, and $\sigma_\theta^2 = \mathrm{var}(\theta_0)$ for all $i, j \in N$ with $i \neq j$ by the symmetry of both payoff and information structures. Thus, without loss of generality, the objective function is written as

$$V' \bullet X' = nv_1\sigma_a^2 + n(n-1)v_2\rho_a\sigma_a^2 + 2nv_3\rho_{a\theta}\sigma_a\sigma_\theta,$$

where

$$V' = \begin{bmatrix} v_1 & v_2 & \cdots & v_2 & v_3 \\ v_2 & v_1 & \cdots & v_2 & v_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_2 & v_2 & \cdots & v_1 & v_3 \\ v_3 & v_3 & \cdots & v_3 & 0 \end{bmatrix}, \; X' = \begin{bmatrix} \sigma_a^2 & \rho_a\sigma_a^2 & \cdots & \rho_a\sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_a\sigma_a^2 & \sigma_a^2 & \cdots & \rho_a\sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_a\sigma_a^2 & \rho_a\sigma_a^2 & \cdots & \sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\theta}\sigma_a\sigma_\theta & \cdots & \rho_{a\theta}\sigma_a\sigma_\theta & \sigma_\theta^2 \end{bmatrix}.$$

Because a payoff structure is of common value, we can use Lemma 2 to rewrite $V' \bullet X'$ as a linear combination of the variance and the covariance of actions:

$$V' \bullet X' = W \bullet X_{11} = nc\sigma_a^2 + n(n-1)d\rho_a\sigma_a^2,$$

where

$$W = \begin{bmatrix} c & d & \cdots & d \\ d & c & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & c \end{bmatrix}$$

with $c = v_1 + 2v_3$ and $d = v_2 + 2v_3 q$. The constraints $X' \in \mathcal{S}_+^{n+1}$, $S \bullet X' = \mathrm{var}(\theta_0)$, and $R'_k \bullet X' = 0$ are reduced to

$$\rho_{a\theta}^2 \leq \frac{n-1}{n}\rho_a + \frac{1}{n}, \tag{26}$$

$$\sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1 - \alpha\rho_a}, \tag{27}$$

where (26) is derived from the Schur complement of X'.

By Proposition 9, if $c \leq d \leq -c/(n-1)$, then no information disclosure is optimal because W is negative semidefinite.[8] By Proposition 10, if $c > 0$ and $d = -\alpha c/(n-1)$, then full information disclosure is optimal because $W = cQ$ and $c > 0$.

We can also characterize the optimal information structure in the other cases by direct calculation. Let $\zeta = nc$ and $\eta = n(c + (n-1)d)$, whereby the objective function is

$$F(\sigma_a, \rho_a) \equiv (\zeta + (\eta - \zeta)\rho_a)\sigma_a^2 = \zeta(1 - \rho_a)\sigma_a^2 + \eta\rho_a\sigma_a^2. \tag{28}$$

If $\zeta = 0$ and $\eta = 1$, then $F(\sigma_a, \rho_a) = \rho_a\sigma_a^2$, so full information disclosure is optimal because the covariance of actions $\rho_a\sigma_a^2$ is maximized when all players know θ_0. In contrast, if $\zeta = 1$ and $\eta = 0$, then $F(\sigma_a, \rho_a) = (1 - \rho_a)\sigma_a^2$, so full information disclosure cannot be optimal because the subtraction of the covariance from the variance of actions $(1 - \rho_a)\sigma_a^2$ is minimized when all players know θ_0. This observation suggests that the optimal information structure can be understood in terms of the ratio of η and ζ as well as their signs, which is confirmed by the following proposition.

Proposition 11. *The optimal information structure is given below.*

(i) *Full information disclosure is optimal if* $\eta > 0$ *and* $\eta \geq n(1-\alpha)\zeta/(2n-1+\alpha)$.

[8]Principle minors of W are $(c + (k-1)d)(c-d)^{k-1}$ for $k \in \{1, \dots, n\}$, so we can verify that W is negative semidefinite if and only if $c \leq d \leq -c/(n-1)$.

(ii) *Partial information disclosure is optimal if* $\zeta > \eta/n$ *and* $\zeta > (2n-1+\alpha)\eta/(n(1-\alpha))$, *where the optimal covariance matrix of* (a, θ_0) *is given by*

$$\rho_a = -((2\alpha+n-2)\zeta+\eta)/(((n-2)\alpha-2(n-1))\zeta+(\alpha+2(n-1))\eta), \tag{29}$$

$$\rho_{a\theta} = \sqrt{(n-1)\rho_a/n+1/n}, \tag{30}$$

$$\sigma_a^2 = \rho_{a\theta}^2\sigma_\theta^2/(1-\alpha\rho_a)^2. \tag{31}$$

(iii) *No information disclosure is optimal if* $\eta \leq 0$ *and* $\zeta \leq \eta/n$.

Proof. We maximize (28) subject to (26) and (27). Plugging (27) into (28), we have

$$F(\sigma_a, \rho_a) = (\zeta + (\eta-\zeta)\rho_a)\left(\frac{\rho_{a\theta}\sigma_\theta}{1-\alpha\rho_a}\right)^2.$$

By (26), $-1/(n-1) \leq \rho_a \leq 1$ and

$$\left(\frac{\rho_{a\theta}\sigma_\theta}{1-\alpha\rho_a}\right)^2 \leq \frac{\sigma_\theta^2}{(1-\alpha\rho_a)^2}\left(\frac{n-1}{n}\rho_a + \frac{1}{n}\right).$$

Thus, by setting

$$f(x) \equiv \sigma_\theta^2\frac{\zeta+(\eta-\zeta)x}{(1-\alpha x)^2}\left(\frac{n-1}{n}x+\frac{1}{n}\right),$$

we obtain

$$F(\sigma_a, \rho_a) \leq \max\{0, f(\rho_a)\} \leq \max_{x\in[-1/(n-1),1]} f(x).$$

Moreover, by (26) and (27), if $\rho_a \in \arg\max_{x\in[-1/(n-1),1]} f(x)$, then $F(\sigma_a, \rho_a) = f(\rho_a)$ with (30) and (31), where $\rho_a = 1$ implies full information disclosure because $\rho_{a\theta} = 1$ by (30), and $\rho_a = -1/(n-1)$ implies no information disclosure because $\sigma_a^2 = 0$ and $\rho_a\sigma_a^2 = 0$ by (31). Therefore, to maximize $F(\sigma_a, \rho_a)$ subject to (26) and (27), it is enough to solve

$$\max_{-1/(n-1)\leq x\leq 1} f(x). \tag{32}$$

The first derivative of $f(s)$ is

$$f'(x) = \frac{\phi(x)}{n(1-\alpha x)^3},$$

where the numerator

$$\phi(x) = (((n-2)\,\alpha - 2\,(n-1))\,\zeta + (\alpha + 2\,(n-1))\,\eta)\,x + (2\alpha+n-2)\,\zeta + \eta$$

is a linear function of x, and the denominator is positive since $-(n-1) < \alpha < 1$ and $-1/(n-1) \leq x \leq 1$.

Suppose that $f'(-1/(n-1)) > 0$ and $f'(1) < 0$, which is true if and only if $\zeta > \eta/n$ and $\zeta > (2n-1+\alpha)\eta/(n(1-\alpha))$ because

$$\phi(-1/(n-1)) = \frac{(n-1+\alpha)(n\zeta-\eta)}{n-1} \text{ and } \phi(1) = -n(1-\alpha)\zeta + (2n-1+\alpha)\eta.$$

In this case, (32) has an interior solution ρ_a with $f'(\rho_a) = 0$ (i.e. $\phi(\rho_a) = 0$), and ρ_a in (29) is the unique solution.

Suppose otherwise. Then, (32) has a corner solution and

$$\max_{-1/(n-1)\leq x\leq 1} f(x) = \max\{f(-1/(n-1)), f(1)\} = \max\left\{0, \frac{\sigma_\theta^2\eta}{(1-\alpha)^2}\right\}.$$

If $\eta > 0$, then $\rho_a = \rho_{a\theta} = 1$ at the corner solution, which implies the optimality of full information disclosure. Because $\eta > 0$ implies $(2n-1+\alpha)\eta/(n(1-\alpha)) > \eta/n$, we must have $\zeta \leq (2n-1+\alpha)\eta/(n(1-\alpha))$. If $\eta \leq 0$, then $\rho_{a\theta} = -1/(n-1)$ at the corner solution, which implies the optimality of no information disclosure. Because $\eta \leq 0$ implies $(2n-1+\alpha)\eta/(n(1-\alpha)) \leq \eta/n$, we must have $\zeta \leq \eta/n$. □

Figure 1 illustrates the optimal information structures, where the horizontal axis is the ζ-axis and the vertical axis is the η-axis. If η is sufficiently large compared to ζ, then full information disclosure is optimal by (i). If ζ is sufficiently large compared to η, then partial information disclosure is optimal by (ii). Otherwise, no information disclosure is optimal by (iii). It is straightforward to show that the condition in (iii) holds if and only if $c \leq d \leq -c/(n-1)$. That is, no information structure is optimal if and only if W is negative semidefinite.

Angeletos and Pavan (2007), Bergemann and Morris (2012, 2013), and Ui and Yoshizawa (2015) study this class of LQG games. Angeletos and Pavan (2007) and Ui and Yoshizawa (2015) focus on information structures in which player's signal consists of an idiosyncratic private signal and a public signal. In particular, Ui and Yoshizawa (2015) obtain the optimal combination of them.[9] On the other hand, Bergemann and Morris (2012, 2013) characterize the set of all symmetric BCE. Moreover, they obtain the optimal information structure in the special case of $\zeta = \eta = 1$, which corresponds the total expected profit in the Cournot game. In contrast, we have characterized the optimal information structures for arbitrary quadratic objective functions in terms of the property of the matrix W and, in particular, the coefficients of the objective functions, ζ and η.

[9]Ui (2016b) considers a symmetric LQG game with endogenous private information acquisition and characterizes the optimal public information disclosure.

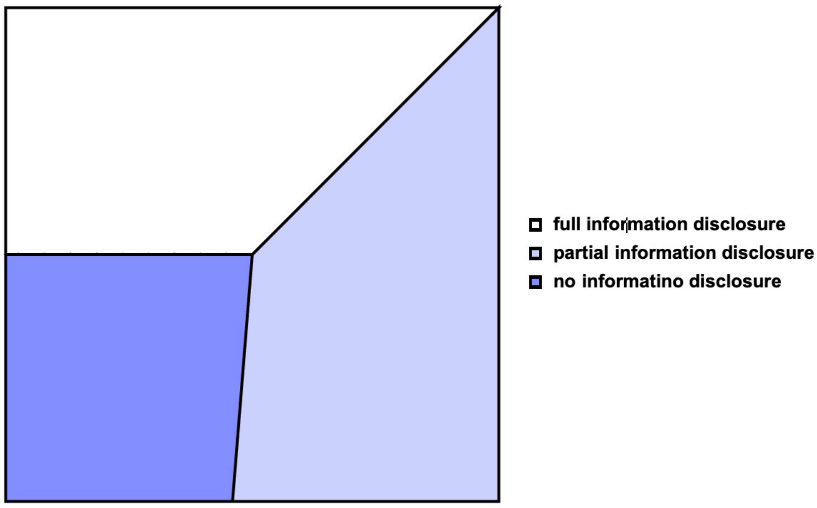

Figure 1: The optimal information structures on the $\zeta\eta$ plane

5 Public information structures

In this section, we study an LQG information design problem when the set of feasible information structures is that of public information structures, i.e., $\Pi = \Pi^p$. As discussed in Section 3.3, it is enough to solve (22).

Recall that D in (22) is an $n \times k$ matrix of rank k such that $\mathrm{var}(\theta) = DD^\top$ and k is the rank of $\mathrm{var}(\theta)$. The matrix $D^\top V_Q D$ is factored as $U\Lambda U^\top$, where $U = [u_1, \ldots, u_k]$ is a $k \times k$ orthogonal matrix and $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_k)$ is a $k \times k$ diagonal matrix with $\lambda_1 \geq \cdots \geq \lambda_m > 0 \geq \lambda_{m+1} \geq \cdots \geq \lambda_k$. Note that $u_1, \ldots, u_k$ are eigenvectors of $D^\top V_Q D$ and $\lambda_1, \ldots, \lambda_k$ are eigenvalues associated with them. We write $U_m = [u_1, \ldots, u_m]$, which is a $k \times m$ matrix consisting of the eigenvectors with strictly positive eigenvalues. The following proposition characterizes the optimal public information structures in terms of U_m.

Proposition 12. *A public signal under the optimal public information structure is given by $t_0 = U_m^\top (D^\top D)^{-1} D^\top \theta$, and the maximum of the objective function equals the sum of the positive eigenvalues $\lambda_1 + \cdots + \lambda_m$. In particular, if $k = n$, then we can choose $D = \mathrm{var}(\theta)^{1/2}$, i.e., the principle square root of* $\mathrm{var}(\theta)$*, so the optimal public signal is $t_0 = U_m^\top \mathrm{var}(\theta)^{-1/2}\theta$.*

Recall Propositions 6 and 7: if $D^\top V_Q D$ is negative semidefinite, i.e., all the eigenvalues are nonpositive, then no information disclosure is optimal, and if $D^\top V_Q D$ is positive semidefinite, i.e., all the eigenvalues are nonnegative, then full information disclosure is optimal. Proposition 12 generalizes Propositions 6 and 7 by identifying the optimal public signal when $D^\top V_Q D$

is neither positive nor negative semidefinite. In particular, it is shown that the maximum of the objective function equals the sum of the positive eigenvalues of $D^\top V_Q D$ and the optimal public signal is constructed from the corresponding eigenvectors $u_1, \dots, u_m$.

To prove Proposition 12, we can directly use the result of Tamura (2017). Tamura (2017) studies a Bayesian persuasion problem (Kamenica and Gentzkow, 2011), or an information design problem with a single player, with a quadratic objective function. In his model, an individual receives a signal t about a state vector $\theta \in \mathbb{R}^n$, and the individual's best response is assumed to be $E[\theta|t] \in \mathbb{R}^n$. The objective function is given by $W \bullet S$, where $W \in \mathcal{S}^n$ is a constant matrix and $S = \mathrm{var}(E[\theta|t]) \in \mathcal{S}^n_+$ is the covariance matrix of the individual's action. Tamura (2017) shows that the maximization of the objective function is reduced to

$$\max_S W \bullet S \text{ s.t. } S \in \mathcal{S}^n_+ \text{ and } \mathrm{var}(\theta) - S \in \mathcal{S}^n_+. \tag{33}$$

He obtains a closed form solution by assuming that $\mathrm{var}(\theta)$ has rank n but without assuming a normal distribution of θ and t. Tamura (2017) also demonstrates that (33) is useful in studying an LQG network game with n players and obtains an optimal public information structure in a special case of an LQG network game when $\mathrm{var}(\theta)$ has full rank n. Thus, Proposition 12 is a generalization of the results of Tamura (2017) where $\mathrm{var}(\theta)$ does not necessarily have rank n (such as a common value payoff structure).

When $\mathrm{var}(\theta)$ has rank n in our LQG information design problem with n players, we can solve (22) and obtain $t_0 = U_m^\top \mathrm{var}(\theta)^{-1/2}\theta$ in Proposition 12 by using the solution of (33) obtained by Tamura (2017). When the rank of $\mathrm{var}(\theta)$ is strictly less than k, however, we must modify the solution of Tamura (2017). Thus, we provide a proof for completeness. The proof is more direct and simpler because we use the properties of normal distributions, which Tamura (2017) does not assume in solving (33).

Proof of Proposition 12. Recall (23) in the proof of Proposition 6. Note that

$$D^\top V_Q D \bullet S = \sum_{l=1}^{k} \lambda_l \gamma_l \leq \sum_{l=1}^{m} \lambda_l \tag{34}$$

because $\lambda_l > 0$ if and only if $l \leq m$ and $0 \leq \gamma_l \leq 1$ for all l. We show that $t_0 = U_m^\top (D^\top D)^{-1} D^\top \theta$ achieves the upper bound $\sum_{l=1}^m \lambda_l$ in (34).

We use the following properties of multivariate normal distributions. When two random vectors $X_1 \in \mathbb{R}^{n_1}$ and $X_2 \in \mathbb{R}^{n_2}$ are jointly normally distributed with $\mathrm{cov}(X_i, X_j) = \Sigma_{ij}$ for $i, j \in$

$\{1,2\}$, the covariance matrix of the conditional expectation of X_2 given X_1 is $\mathrm{var}(E[X_2|X_1]) = \Sigma_{21}(\Sigma_{11})^{-1}\Sigma_{12}$. Using this, we can verify that $\mathrm{var}(E[\theta|t_0]) = DU_m(U_m^\top U_m)^{-1}U_m D^\top$ and

$$\begin{aligned} V_Q \bullet \mathrm{var}(E[\theta|t_0]) &= \mathrm{tr}\,(V_Q D U_m (U_m^\top U_m)^{-1} {U_m}^\top D^\top) \\ &= \mathrm{tr}\,(D^\top V_Q D U_m {U_m}^\top) \\ &= \mathrm{tr}\,(U\Lambda U^\top U_m {U_m}^\top) \\ &= \mathrm{tr}\,(\Lambda U^\top U_m {U_m}^\top U) = \sum_{l=1}^{m} \lambda^l. \end{aligned}$$

The last equality holds because $U^\top U_m = [\delta_{ij}]_{k\times m}$, where δ_{ij} is the Kronecker delta. □

Proposition 12 has the following implication for optimality of partial information disclosure in Π^* when $D^\top V_Q D$ is neither negative semidefinite nor positive semidefinite, which complements Corollary 8.

Corollary 13. *If $D^\top V_Q D$ is neither negative semidefinite nor positive semidefinite, then partial information disclosure is optimal in* Π^*.

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