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Decentralizability of efficient allocations with heterogenous forecasts

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Abstract

Do price forecasts of rational economic agents need to coincide in perfectly competitive complete markets in order for markets to allocate resources efficiently? To address this question, we define an efficient temporary equilibrium (ETE) within the framework of a two period economy. Although an ETE allocation is intertemporally efficient and is obtained by perfect competition, it can arise without the agents forecasts being coordinated on a perfect foresight price. We show that there is a one dimensional set of such Pareto efficient allocations for generic endowments.

JEL classification numbers: D51, D53, D61

1 Introduction

Intertemporal trade in complete markets is known to achieve Pareto efficiency when the price forecasts of agents coincide and are correct. The usual justification for this coincidence of price forecasts is that if agents understand the market environment perfectly, they ought to reach the same conclusions, and hence in particular, their forecasts must coincide. But it is against the spirit of perfect competition to require that agents

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should understand the market environment beyond the market prices they commonly observe; we therefore study intertemporal trade without requiring that price forecasts of heterogenous agents coincide.

To address this issue precisely, we study a sequence of commodity markets with no uncertainty, where there is a riskless bond market so that markets are complete. Specifically, we consider a two period (periods 0 and 1 respectively) pure exchange economy with at least two households, with finitely many perishable commodities in each period, and a riskless bond that pays in period 1 dollars. We ask what Pareto efficient allocations can be decentralized by a Walrasian model that respects the intertemporal structure, i.e., there be competitive spot markets for each period for the consumption goods available in that period, and a competitive market for the bond in period 0.

In period 0 each household optimizes given spot prices and the bond price observed in period 0, with its price forecast for the period 1 spot prices. The period 0 spot prices and the bond price are determined to clear the markets in period 0. The price forecasts of different households are allowed to be heterogenous. Given the savings of the households from period 0, the period 1 spot prices emerge to clear the commodity markets in period 1; these market clearing spot prices will be in general different from the heterogenous forecasts made by the agents in period 0. The resulting equilibrium is referred to as a temporary equilibrium.

In this set up, if one *assumes* that the price forecasts of all agents coincide and agree with the period 1 market clearing prices, the resulting temporary equilibrium is referred to as a perfect foresight equilibrium (PFE). With the bond market, the markets under perfect foresight are complete, the ensuing equilibrium allocation coincides with an Arrow Debreu (henceforth AD) allocation and is Pareto efficient by the first fundamental theorem of welfare economics. This is of course a classic result formalized by Arrow (1964) and then elaborated by Radner (1972). Thus under perfect foresight, the AD allocations are the only ones that can be decentralized as Walrasian (temporary) equilibria. By the theorem of Debreu (1970), there are finitely many AD allocations, generically in endowments. To summarize, generically in endowments, the set of Pareto efficient allocations that can be decentralized as Walrasian equilibria with perfect foresight is zero dimensional.

The PFE approach explains prices and is able to address welfare issues, but it incurs a serious cost in that perfect foresight is assumed, rather than derived. As is expressed by various scholars, the assumption of perfect foresight is extraordinarily strong; a case in point is Radner's own critique of perfect foresight.[1] It goes without saying that this approach is absolutely inadequate for comparing the quality of price forecasts and explaining, among other issues, the use of policy tools that seek to influence the forecasts of diverse subsets of agents. In spite of these obvious shortcomings, the pervasive use of this approach would appear to stem from the presumption that perfect foresight is indispensable to a market theory that delivers efficient outcomes and retains some predictive power.

The following classical question on price forecasts therefore seems a very natural one to pose in this setup: First, require that all the spot markets clear in the temporary equilibrium sense. That is, even when the households traded anticipating wrong prices in the past, describe how they consume and save competitively in every period so that one can address welfare issues. Secondly, suppose that the underlying trading processes are so elaborated that the resulting sequence of consumption constitutes a Pareto efficient allocation, not only within each period but also intertemporally. However, market clearing and the efficiency property of the allocation does not rule out forecasts that are inadequate in that they lead to regret. We rule out such time inconsistent forecasts by imposing as our final requirement a retrospective consistency condition on forecasts, and propose an efficient temporary equilibrium with retrospective consistency (henceforth, ETEC) as our solution concept. An allocation arising from an ETEC is by construction decentralized by market prices. The question we pose is, must an ETEC necessarily be a perfect foresight equilibrium?

At first sight the answer might appear positive, under the standard set of assumptions

[1]On page 942, Radner (1982) writes "Although it is capable of describing a richer set of institutions and behaviour than is the Arrow-Debreu model, the perfect foresight approach is contrary to the spirit of much of competitive market theory in that it postulates that individual traders must be able to forecast, in some sense, the equilibrium prices that will prevail in the future under all alternative states of the environment. Even if one grants the extenuating circumstances mentioned in previous paragraphs, this approach still seems to require of the traders a capacity for imagination and computation far beyond what is realistic."

on utility functions such as monotonicity, concavity, and differentiability. Intuitively, the dimension of Pareto efficient allocations should be one less than the number of the households, since it is in effect the set of wealth transfers across households. On the other hand, at an ETEC, since the final consumption bundle must be attained in markets, each household's consumption bundle must satisfy some budget constraint. By market clearing one of these budget constraints might be redundant, but still these create additional restrictions at least as many as the dimension of Pareto efficient allocations. Recall that the set of Arrow-Debreu equilibrium allocations can be found from Pareto efficient allocations and budget constraints by the second fundamental theorem of welfare economics, and Debreu's generic finiteness theorem shows that the set of Arrow-Debreu equilibria is zero dimensional generically. Therefore, the same logic seems to suggest that the set of ETEC allocations is zero dimensional, at least generically. Hence if an ETEC which does not entail perfect foresight ever exists, it must be an isolated case relying on some coincidence.

The surprise, the aforementioned logic notwithstanding, is that this conjecture is incorrect. More precisely, our main result shows the existence of a one dimensional set of ETEC allocations around each Arrow-Debreu equilibrium allocation, generically in endowments whenever the utilities of households are time separable. To clarify the role of the assumption of time separability and to simplify our analysis, we first introduce the notion of a Quasi-ETEC, which is obtained by relaxing the role of forecasts in an ETEC and is a necessary condition for an ETEC. We first establish a generic indeterminacy result for Quasi-ETEC for general utility functions. We then impose time separability for establishing the equivalence of Quasi-ETEC with ETEC.

Curiously enough, the degree of real indeterminacy does not depend on the number of households, while the dimension of Pareto efficient allocations increases with the number of households as explained above. Therefore, when the number of households is very large, which is a plausible circumstance for perfect competition, an ETEC does require a very delicate alignment of price forecasts. If one conjectured, despite our intuitive illustration using budget constraints, that an ETEC would hardly restrict price forecasts, then the invariance to the number of households should turn up as a surprising result.

Coming back to the question we posed above, namely whether or not an ETEC is necessarily an AD equilibrium, our answer is that decentralized markets are able to deliver a significantly larger set of acceptable (Pareto efficient) outcomes under less restrictive assumptions on forecasts. Moreover, the extra degree of freedom due to heterogeneity of forecasts is only one at least in our model, so the explanatory power is almost as strong as the perfect foresight approach, marking a stark contrast with the classical temporary equilibrium literature (e.g., Grandmont (1972)), which assumes price forecasts rather than derives, and hence suffers from lack of explanatory power. Therefore, we contend that the approach based on ETEC has considerably greater descriptive appeal than believed erstwhile. In this context, it is important to note that ETEC does not rule out some agreement among households regarding future prices: indeed for time separable preferences, an ETEC can be sustained with households forecasts agreeing, and being correct, on second period relative prices but disagreeing on the inflation rate up to one degree of freedom, under a mild regularity condition.

We interpret our existence result for ETEC as a decentralization theorem since it shows that Walrasian markets can lead the economy to a (one dimensional) set of Pareto efficient allocations. Our notion of decentralization differs from the classical second welfare theorem approach in one crucial aspect, namely, that we do not require lump sum transfers to be imposed by the planner. Indeed as the literature on implementation and incentives emphasizes, the use of lump sum transfers in a decentralization story is problematic as agents have to be incentivized to reveal their true preferences and endowments. In our set up, these transfers are implied by the structure of forecasting errors induced in an ETEC, and can be summarized by the discrepancy between the realized inflation rate in the market and the forecasted inflation rates of the households. An attractive feature of our notion of decentralization is thus that the requisite transfers arise endogenously without a planner's explicit intervention, and trade is completely voluntary and anonymous. So an implication of our result is that lump sum transfers might occur through self selecting market transactions, up to exactly one degree of freedom.

The paper is organized as follows. Section 2 specifies the model and the Definitions. Section 3 introduces the key notion of a Quasi-ETEC. Section 4 provides a characterization result of Quasi-ETEC. Section 5 establishes the generic indeterminacy of Quasi-ETEC while Section 6 proves our main result on the indeterminacy of ETEC. Finally, section 7 discusses some extensions of our results, mentions some directions for further work, and related literature.

2 The Model and Definitions

We consider a standard competitive exchange economy with inside money. There are two periods, period 0 and 1, and there are $L_t \geq 1$ perishable consumption goods in each period, $t = 0, 1$, to be traded competitively. Write $L = L_0 + L_1$.

There are $H \geq 1$ households, labelled by $h = 1, ..., H$. Abusing notation we use H for the set of households as well. Household h is endowed with a vector e_h^0 of goods in the first period (period 0) and a vector e_h^1 of goods in the second period (period 1). We write $e_h = \left(e_h^0, e_h^1\right) \in \mathbb{R}^{L_0} \times \mathbb{R}^{L_1}$.

Household h's consumption set is $X_h = \mathbb{R}_+^{L_0} \times \mathbb{R}_+^{L_1}$, with a generic consumption bundle written as $x_h = \left(x_h^0, x_h^1\right)$. Let $X := \times_{h=1}^H X_h$. Household h's preferences for consumption bundles are represented by an increasing utility function $u_h : X_h \to \mathbb{R}$. Later, we will make assumptions on u_h so that consumption takes place in the interior of X_h.

In the first period, a bond which pays off $1 + r$ $(r > -1)$ units in units of account (dollar) in the second period is traded competitively, i.e., a household takes the market interest rate r as given to decide its saving. A negative saving corresponds to borrowing. There is no uncertainty and no limit for saving and borrowing. The net supply of the bond is zero, so it is inside money whose real return is determined in the markets. Writing z_h for the amount of saving of household h, and writing $p^0 \in \mathbb{R}_+^{L_0}$ for the market prices of the consumption goods in period 0, the consumption bundle x_h^0 of household h in period 0 is therefore subject to

$$p^0 \cdot x_h^0 + z_h \leq p^0 \cdot e_h^0. \tag{1}$$

There is no futures market which might help predict the prices of the consumption goods in the second period, and hence we do not impose perfect foresight about market prices in future a priori. Rather, we assume that each household h first anticipates the prices $\hat{p}_h \in \mathbb{R}_+^{L_1}$ of the goods in period 1 in order to decide consumption and saving/borrowing in period 0. We shall also refer to $\hat{p}_h$ as the forecast of household h. Default is not allowed in our model: that is, no household plans on defaulting given his forecast. Then, at the prevailing market interest rate r, household h expects that his period 1 consumption bundle $\hat{x}_h^1$ must meet the period 1 budget

$$\hat{p}_h \cdot \left(\hat{x}_h^1 - e_h^1\right) \leq (1+r)\, z_h \tag{2}$$

if his saving is z_h. Since there is no limit for saving/borrowing with no default in our model, by eliminating z_h from (1) and (2), household h faces in effect the following budget constraint for consumption goods when it determines period 0 consumption:

$$p^0 \cdot \left(x_h^0 - e_h^0\right) + \frac{1}{1+r}\hat{p}_h \cdot \left(\hat{x}_h^1 - e_h^1\right) \leq 0. \tag{3}$$

It is readily seen that if $\left(x_h^0, \hat{x}_h^1\right) \in X_h$ satisfies (3), then there is z_h with which the budget is met in both periods. So it appears as if household h has a consumption plan $\hat{x}_h^1$ for period 1, in addition to forecast $\hat{p}_h$, when the period 0 consumption bundle is chosen. Note that the monotonicity of u_h will assure that the equality will hold at the optimum in (3).

We denote the market prices of the goods in period 1 by $p^1 \in \mathbb{R}_+^{L_1}$. That is, in period 1, household h is subject to the constraint

$$p^1 \cdot \left(x_h^1 - e_h^1\right) \leq (1+r)\, z_h, \tag{4}$$

i.e., the market value of the net consumption must be no greater than the nominal return from the saving. Notice that z_h is already determined before period 1 markets open. In conclusion, the realized consumption path $\left(x_h^0, x_h^1\right)$ must satisfy the following equation:

$$p^0 \cdot \left(x_h^0 - e_h^0\right) + \frac{1}{1+r}p^1 \cdot \left(x_h^1 - e_h^1\right) \leq 0. \tag{5}$$

Note that although constraint (5) is not taken into account in period 0, household h will spend all the income in period 1 at the market price, i.e., $p^1 \cdot \left(x_h^1 - e_h^1\right) = (1+r)\, z_h$ will hold if u_h is increasing, and then the equality holds for (5) at the optimum.

Now we shall define a dynamic temporary equilibrium: it is simply the standard classical temporary equilibrium notion applied for each period.

Definition 1 *A temporary equilibrium (TE) is a tuple* $\left(x^*, (\hat{p}_h)_{h=1}^H, r^*, \left(p^{0*}, p^{1*}\right)\right) \in X \times \left(\mathbb{R}_+^{L_1}\right)^H \times (-1, \infty) \times \left(\mathbb{R}_+^{L_0+L_1}\right)$ *such that:*
(i) x^* *is a feasible allocation, i.e.,* $\sum_{h=1}^H x_h^* = \sum_{h=1}^H e_h$*;*
(ii) for each $h \in H$*, there exists* $\hat{x}_h^1$ *such that* $\left(x_h^{0*}, \hat{x}_h^1\right)$ *maximizes utility in consumption set* X_h *under budget (3) given* $\hat{p}_h$ *at* $r = r^*$ *and* $p^0 = p^{0*}$*;*
(iii) for each $h \in H$*,* x_h^{1*} *maximizes* $u_h\left(x_h^{0*}, \cdot\right)$ *in* $\mathbb{R}_+^{L_1}$ *under constraint (5) at* $r = r^*$*,* $p^0 = p^{0*}$ *and* $p^1 = p^{1*}$*, and* $x_h^0 = x_h^{0*}$*.*

Note that condition (i) implies that the total demand meets the total supply in every market in both periods. Then, condition (ii) says that period 0 markets are in *temporal equilibrium* given forecasts $(\hat{p}_h)_{h=1}^H$, and condition (iii) says that the period 1 markets are also in temporal equilibrium, given the consumption allocation in period 0.

Remark 2 *There is an obvious nominal indeterminacy due to the homogeneity of (3) and (5): if* $\left(x^*, (\hat{p}_h)_{h=1}^H, r^*, \left(p^{0*}, p^{1*}\right)\right)$ *is a TE, so are* $\left(x^*, (\hat{p}_h/(1+r^*))_{h=1}^H, 0, \left(p^{0*}, p^{1*}/(1+r^*)\right)\right)$ *and* $\left(x^*, (t\hat{p}_h)_{h=1}^H, r^*, \left(tp^{0*}, tp^{1*}\right)\right)$ *for any* $t > 0$*. The homogeneity of (3) shows that, as far as temporal equilibrium allocations with positive prices are concerned, there is no loss of generality if we focus on a temporal equilibrium with* $r^* = 0$*, i.e., the nominal interest rate is zero. So from now on, we always normalize the interest rate equal to zero, and refer to a TE as a tuple* $\left(x^*, (\hat{p}_h)_{h=1}^H, \left(p^{0*}, p^{1*}\right)\right)$*. The homogeneity of (5) then shows that one may normalize one of the market prices equal to one in addition.*[2]

Figure 1 describes a household's problem for the simplest case of $L_0 = L_1 = 1$. Notice that consumption bundle $\left(x^0, \hat{x}^1\right)$ is utility maximizing given forecast $\hat{p}^1$ and that the realized consumption path $\left(x^0, x^1\right)$ must respect the budget constraint with realized market price p^1. In this simplest case, since period 1 trade is trivial, it appears as if the household is forced to choose x^1 although $\left(x^0, x^1\right)$ is not necessarily utility maximizing.

There is hardly any restriction on equilibrium forecasts besides various possibilities of price normalization, and hence there are many temporary equilibria. Since the marginal rates of substitution of a pair of goods in different periods are not necessarily equated

[2] One could choose a different normalization, for instance, setting one of the prices equal to one for each of the two periods, and keep the interest rate as an equilibrating variable.

among agents, a temporary equilibrium tends not to be intertemporally efficient.[3] But if one subscribes to the hypothesis that a perfect market structure as a whole would induce the households to trade until gains from trade vanish completely from their viewpoints, it is natural to focus on an efficient temporary equilibrium. Even without such an extreme view, since an efficient allocation can be decentralized in competitive markets only when it constitutes a temporary equilibrium, an efficient temporary equilibrium is readily seen as an important benchmark.

Definition 3 *An efficient temporary equilibrium (ETE) is a temporary equilibrium* $\left(x^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ *where the consumption allocation* x^* *is Pareto efficient.*

The extreme instance of an ETE is a *perfect foresight equilibrium* (henceforth, PFE): by definition, a PFE is a particular temporal equilibrium $\left(x^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ where $\hat{p}_h = p^{1*}$ for all h, i.e., in period 0, each household correctly forecasts the period 1 market prices to be realized. In this case, the two budget constraints (3) and (5) are identical, and each household's utility must be maximized within the common budget set. Hence a PFE is an *Arrow-Debreu equilibrium* (AD equilibrium) where any temporal good can be traded, and vice versa. Thus we shall use PFE and AD equilibrium interchangeably depending on the context. Needless to say, an Arrow-Debreu equilibrium is weakly efficient, and if utility functions are continuous and increasing, it is Pareto efficient by the first fundamental theorem of welfare economics. So under the standard assumptions, a PFE is an ETE. Our focus will therefore be on the question of whether or not efficiency induces perfect foresight.

While a hypothetical market transaction process justifying an ETE would rule out many forecasts which would allow unrealized gains from trade, some low quality forecasts might survive in an ETE by chance nonetheless. To see this, notice that if the planned consumption $\hat{x}_h$ is based on a very inaccurate forecast, it might be very different from the realized consumption x_h^{*1}. Then although the consumption allocation is efficient and thus there are no gains from trade, household h may regret the consumption of x_h^{0*} at

[3]Any temporary equilibrium satisfies the property that the second period allocation is efficient conditional on a given fixed allocation of the first period. We require here instead Pareto efficiency of the entire intertemporal allocation.

period 0 market prices and might wish to engage in additional trading at those prices if possible.

On the other hand, if household h correctly anticipated x_h^{1*} in period 0 under guidance of a good forecast, then there would be no incentive for re-trading and consequently no regret. This observation provides a rationale for the quality of the price forecasts in an ETE, which we formalize as follows.

Definition 4 *An ETE with retrospective consistency (ETEC) is an ETE* $\left(x^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ *such that for every* $h = 1, ..., H$, $u_h\left(x_h^0, x_h^{*1}\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$ *implies* $p^{*0} \cdot x_h^0 > p^{*0} \cdot x_h^{*0}$.

Since at a PFE all the households correctly anticipate the future consumption, a PFE trivially exhibits retrospectively consistency and therefore it is an ETEC. While at first sight the retrospective consistency condition introduced above may appear too stringent, we observe below that it is automatically satisfied if utility functions are (non-linear) time separable:

Definition 5 *Utility function* u_h *is said to be time separable if* $u_h\left(x_h^0, x_h^1\right) = W_h\left(u_h^0\left(x_h^0\right), u_h^1\left(x_h^1\right)\right)$ *where* $u_h^t : \mathbb{R}_+^{L_t} \to \mathbb{R}$, $t = 0, 1$, *are increasing and* W_h *is increasing.*

Note that if $L_0 = L_1 = 1$, i.e., there is only one good in period 0, then utility functions are trivially time separable. With time separability for all households, an ETE exhibits retrospective consistency and is thus an ETEC; indeed $u_h\left(x_h^0, x_h^{*1}\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$ implies $u_h^0\left(x_h^0\right) > u_h^0\left(x_h^{*0}\right)$ since W_h is increasing, and so $p^{*0} \cdot x_h^0 > p^{*0} \cdot x_h^{*0}$ must hold, by the utility maximization in period 0 markets required for temporary equilibrium and the monotonicity of u_h^0.

While it may be of theoretical interest to investigate the implications of efficiency on time inconsistent forecasts, we confine attention in this paper to ETEC. This extra consistency requirement about the quality of forecasts puts more structure on our study and appears appropriate since it addresses a long standing criticism of the temporary equilibrium approach, namely, that it puts very little structure on forecasts.

One might expect that the efficiency and the retrospective consistency (or the time separability requirement) are so stringent that an ETEC needs to be a PFE. That is, efficiency and consistency imply a common and correct forecast. At this point it is useful to provide a simple graphical example of $L_0 = L_1 = 1$ and $H = 2$, which suggests that this assertion must be false. In Figure 2, an arbitrary allocation (x_1, x_2) is first chosen from the set of efficient allocations. Then a market price p^1 is found so that each x_h meets the realized budget. Then forecast $\hat{p}_h^1$ is chosen for each h, so that h is willing to consume x_h^0. Assuming that the period demand responds well enough to the forecast, such a $\hat{p}_h^1$ can be readily found. Then by construction, we have found an ETEC since the utility functions are trivially time separable because of a single good in each period.

The graphical argument above might then suggest that any efficient allocation can be a ETEC as long as the period 0 demand is responsive to forecast. Such a conjecture might be reinforced if one recalls that an efficient allocation can always be supported by prices by the second fundamental theorem of welfare economics. It is however, incorrect: we will show that generically in endowments, there is at most a one dimensional manifold of ETEC allocations around any PFE for general utility functions. When utilities are time separable, every element of this candidate one dimensional manifold can be sustained as an ETEC, generically in endowments. A formal statement will be provided after we describe assumptions on utility functions and endowments, which are fairly standard in the literature of general equilibrium with rational expectations.

The restriction of time separability, apart from its obvious decision theoretic and analytical appeal, sits well with our ETEC solution concept. For one, it automatically fulfills the retrospective consistency embodied in an ETEC. Furthermore, our analysis has a bearing on the set of possible ETEC for the case of for general utilities that are not time separable, which we shall illustrate in the concluding section.

3 Role of Forecasts and an Allocation Based Definition

Note that, even with price normalization (Remark 2), different forecasts might induce the same consumption in period 0, generating a large degree of welfare irrelevant indeterminacy, which causes mathematical nuisances. Also, forecasts affect welfare and thus efficiency only through actual consumption. Hence it is analytically more convenient to consider an auxiliary concept focusing on the realized consumption allocation and prices, suppressing unobservable private forecasts. This approach has an additional advantage of not requiring consistency of forecasts, which enables us to work with general utility functions.

For the purpose of suppressing forecasts, we first ask if a period 0 consumption bundle can arise at prevailing period 0 prices, from some forecast and consumption plan. The following rephrases the utility maximization condition for an ETEC from this perspective.

Definition 6 *A consumption vector $x_h^0 \in \mathbb{R}_+^{L_0}$ is said to be a justifiable demand for household h at given prices $p^0 \in \mathbb{R}_{++}^{L_0}$, if there is a forecast $\hat{p}_h \in \mathbb{R}_+^{L_1}$ and a consumption plan $\hat{x}_h \in \mathbb{R}_+^{L_1}$ such that $\left(x_h^0, \hat{x}_h^1\right)$ maximizes u_h under budget $p^0 \cdot \left(x_h^0 - e_h^0\right) + \hat{p}_h \cdot \left(x_h^1 - e_h^1\right) \leq 0$.*

Thus a consumption vector x_h^0 is justifiable at some prices p^0 if it belongs to the projection of the "offer curve" onto period 0 consumption. Note that a consumption vector might never be justifiable at any prices since the endowments are exogenously given: consider the following simple example.

Example 7 *$L_0 = L_1 = 1$, $u_h\left(x^0, x^1\right) = \ln x^0 + \ln x^1$, and $e_h = (1, 0)$. It is readily verified that the demand for good 0 is $\frac{1}{2}$ irrespective of prices. Thus x_h^0 is justifiable at some prices only if $x_h^0 = \frac{1}{2}$.*

The demand for good 0 is constant in the example because the price effect and the (net) income effect on demand for good 0 cancel out at any prices. Although this cancellation does not occur if $e_h >> 0$, the utility function can be suitably modified so that justifiability fails even for some strictly positive endowments. But intuitively, the cancellation of this kind must be coincidental, and so failure of justifiability appears to be non-generic in endowments. Later, we will formalize this idea to establish the generic existence result.

We shall present the auxiliary concept: it is ETEC without justifiability:

Definition 8 *A tuple $\left(x^*, \left(p^{0*}, p^{1*}\right)\right) \in X \times \left(\mathbb{R}_+^{L_0+L_1}\right)$ is said to be a Quasi ETEC if:*
(i) x^* is an efficient allocation;
(ii)' for each $h \in H$, $u_h\left(x_h^0, x_h^{*1}\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$ implies $p^{*0} \cdot x_h^0 > p^{*0} \cdot x_h^{*0}$

(iii)' for each $h \in H$*,* $p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) + p^{*1} \cdot \left(x_h^{*1} - e_h^1\right) = 0$ *holds, and* $u_h\left(x_h^{*0}, x_h^1\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$ *implies* $p^{*1} \cdot \left(x_h^1 - e_h^1\right) > -p^{*0} \cdot \left(x_h^{*0} - e_h^0\right)$*.*

Condition (iii)' above is another way to say x_h^{*1} is utility maximizing subject to the period 1 budget since utility functions are increasing, and so it is equivalent to condition (iii) in Definition 1. So a Quasi-ETEC obtains when period 0 maximization condition (ii) in Definition 1 is replaced with retrospective consistency (ii)', in addition to the efficiency requirement of Definition 3. Recall that retrospective consistency (Definition 4) is implied by the period 0 maximization in some cases, but not vice versa. Thus, an ETEC allocation must be a Quasi-ETEC allocation and a Quasi-ETEC constitutes an ETEC if the period 0 maximization is satisfied with some forecasts, i.e., the period 0 consumption bundle is justifiable. For later reference, we state this trivial observation formally below:

Lemma 9 *A Quasi-ETEC* $\left(x^*, \left(p^{0*}, p^{1*}\right)\right)$ *is an ETEC with some forecasts if and only if* x_h^{0*} *is justifiable at* p^{0*} *for every* $h \in H$*.*

Remark 10 *Just as ETE or ETEC, there is an obvious nominal indeterminacy due to the homogeneity of (ii)' and (iii)', and one can normalize one of the prices equal to one.*

4 Characterization of Quasi-ETEC

Our ultimate goal is to show the real indeterminacy of ETEC consumption allocations around a PFE consumption allocation for a generic set of economies with time separable utility functions. In preparation, we shall first study the structure of Quasi-ETEC allocations around a locally unique PFE allocation, without the time separability assumption in this section. We choose to proceed in this manner in order to clarify the essence of the whole problem, in particular, the role of time separability.

In order to employ the standard technique of genericity analysis, we assume the following: for every household $h = 1, ..., H$,

- u_h is C^2 on $\mathbb{R}^L_{++}$, $\partial u_h \gg 0$, and differentiably strictly quasi-concave[4], and each indifference curve is closed in $\mathbb{R}^L$;

[4] That is, for any $v \in \mathbb{R}^L \setminus 0$ such that $\partial u_h (x) \cdot v = 0$, $v^T \partial^2 u_h (x)\, v < 0$.

- $e_h >> 0$.

We fix utility functions throughout, and identify an economy with its initial endowments: so write $\mathcal{E} := \left(\mathbb{R}^L_{++}\right)^H$ and its generic element is denoted by $e = (\cdots, e_h, \cdots)$. A subset of $\mathcal{E}$ is said to be *generic* if it is open and its complement has Lebesgue measure 0. More generally, for subsets $V \subseteq V'$ of $\mathcal{E}$, we say that V is *generic* in V' if it is open in $\mathcal{E}$ and its relative complement $V' \backslash V$ has Lebesgue measure 0.

Taking advantage of the differentiable structure, we will obtain a dual representation result of Quasi- ETEC. For this purpose, we first recall a standard result from convex analysis, which may be seen as an instance of the familiar Kuhn Tucker condition.[5]

Lemma 11 *Let $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function defined around $x \in \mathbb{R}^n$ which is differentiably strictly quasi-concave. Then the following two statements about $q \in \mathbb{R}^n$ are equivalent: (1) if $f(x') > f(x)$ then $q \cdot x' > q \cdot x$; (2) there is $\alpha > 0$ such that $q = \alpha \partial f(x)$.*

Condition (1) says that $x' = x$ maximizes $f(x')$ subject to $q \cdot x' \leq q \cdot x$, and condition (2) says that the gradient at x is proportional to "price vector" q, i.e., the marginal rate of substitution is equated with the corresponding relative price in the language of consumer theory.

Also we shall use the following dual characterization of an efficient allocation of L $(= L_0 + L_1)$ goods, which is nothing but the fundamental theorems of welfare economics.

Lemma 12 (fundamental theorems of welfare economics) *Let $x = (\cdots, x_h, \cdots) >> 0$ be a feasible allocation. Then the following three conditions are equivalent: (1) x is efficient; (2) there are $\lambda_h > 0$, $h = 1, 2, ..., H$, and a vector $\bar{p} \in \mathbb{R}^L_{++}$ such that $\lambda_h \bar{p} = \partial u_h(x_h)$ holds for all h; (3) there is a vector $\bar{p} \in \mathbb{R}^L_{++}$ and transfers w_h, $h = 1, 2, ..., H$, with $\sum_{h=1}^H w_h = 0$ such that each x_h maximizes $u_h(z)$ given $\bar{p} \cdot (z - e_h) \leq w_h$.*

We first report a clean dual characterization of an Quasi-ETEC.

Proposition 13 *Let x^* be an efficient allocation. Then $\left(x^*, \left(p^{0*}, p^{1*}\right)\right)$ is a Quasi-ETEC if and only if the following two conditions hold:*

[5] Lemma 11 and 12 are standard and we shall omit proofs. See, for instance, Mas-Colell (1985).

(1) there exist $\gamma_h > 0$*,* $h = 1, ..., H$*, and* $\beta > 0$*, such that (a)* $\gamma_h p^{0*} = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0}$ *for each* $h = 1, ..., H$*, and (b)* $\beta\gamma_h p^{1*} = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^1}$ *for each* $h = 1, ..., H$*;*
(2) $p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) + p^{*1} \cdot \left(x_h^{*1} - e_h^1\right) = 0$ *for each* $h = 1, ..., H$*;*

Proof. Suppose there are $\gamma_h > 0$, $h = 1, ..., H$, and $\beta > 0$ which satisfy conditions *(1)* holds, and and also condition *(2)* holds. We need to confirm verify (ii)' and (iii)' of Definition 8.

Condition (ii)' is satisfied: by Lemma 11, condition (a) implies that whenever $u_h\left(x_h^0, x_h^{*1}\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$, $\gamma_h p^{*0} \cdot x_h^0 > \gamma_h p^{*0} \cdot x_h^{*0}$ and hence $p^{*0} \cdot x_h^0 > p^{*0} \cdot x_h^{*0}$. holds.

Condition (iii)' is satisfied: by Lemma 11, condition (b) implies that whenever $u_h\left(x_h^{*0}, x_h^1\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$, $\beta\gamma_h p^{*1} \cdot \left(x_h^1 - e_h^1\right) > \beta\gamma_h p^{*1} \cdot \left(x_h^{*1} - e_h^1\right)$ and hence $p^{*1} \cdot \left(x_h^1 - e_h^1\right) > p^{*1} \cdot \left(x_h^{*1} - e_h^1\right)$ holds. Thus from condition *(2)*, whenever $u_h\left(x_h^{*0}, x_h^1\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$, we have $p^{*1} \cdot \left(x_h^1 - e_h^1\right) > -p^{*0} \cdot \left(x_h^{*0} - e_h^0\right)$.

Conversely, suppose that $\left(x^*, \left(p^{0*}, p^{1*}\right)\right)$ is a Quasi- ETEC. Then, *(2)* holds trivially from (iii)', so it remains to show that there are γ_h and β required for condition *(1)*.

First of all, since x^* is efficient, by the second fundamental theorem of welfare economics (Lemma 12), there are $\lambda_h > 0$, $h = 1, 2, ..., H$ and a vector $\bar{p} = \left(\bar{p}^0, \bar{p}^1\right) \in \mathbb{R}^L$ such that $\lambda_h \bar{p} = \partial u_h\left(x_h^*\right)$, i.e., both $\lambda_h \bar{p}^0 = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0}$ and $\lambda_h \bar{p}^1 = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^1}$ hold for $h = 1, ..., H$.

We shall first construct γ_h for each household h to meet condition (a). Since $u_h\left(x_h^0, x_h^{*1}\right) > u_h\left(x_h^{*0}, x_h^{*1}\right)$ implies $p^{*0} \cdot x_h^0 > p^{*0} \cdot x_h^{*0}$ by (ii)', p^{*0} must be proportional to $\frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0}$ by Lemma 11. Thus for each $h = 1, ..., H$, we can find $\gamma_h > 0$ such that

$$\gamma_h p^{0*} = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0}. \tag{6}$$

Next, we shall find β such that (b) holds. Notice that (6) also implies that p^{*0} must be proportional to $\bar{p}^0$, i.e., $\alpha^0 p^{*0} = \bar{p}^0$ for some $\alpha^0 > 0$, since both p^{0*} and $\bar{p}^0$ are proportional to $\frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0}$ for every h, thanks to the efficiency of x^*. Then from

$$\gamma_h p^{*0} = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0} = \lambda_h \bar{p}^0 = \lambda_h \alpha^0 p^{*0},$$

we deduce that

$$\gamma_h = \lambda_h \alpha^0 \tag{7}$$

holds for each $h = 1, ..., H$. Pick any h for reference, and conclude that p^{*1} must be proportional to $\frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^1} = \lambda_h \bar{p}^1$ from utility maximization condition (iii)' for this h. Consequently, we can find some $\alpha^1 > 0$ such that

$$\frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^1} = \lambda_h \bar{p}^1 = \lambda_h \alpha^1 p^{*1}, \tag{8}$$

holds for each $h = 1, ..., H$.

Set $\beta = \alpha^1/\alpha^0 > 0$. For each household h,

$$\begin{aligned} \beta\gamma_h p^{1*} &= \beta\lambda_h\alpha^0 p^{1*} \text{ (by (7)} \\ &= \lambda_h\alpha^1 p^{1*} \text{ (construction of } \beta) \\ &= \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^1}, \text{ (by (8))} \end{aligned}$$

and so condition (b) is established as we wanted. ■

Remark 14 *Recall one may normalize the price of the first good in period 0 equal to one (see Remarks 2 and 10) In the argument above with such normalization we have* $p^{*0} = \bar{p}^0$, *and* $\beta p^{*1} = \bar{p}^1$ *for some* β.

Observe that condition *(1)* says in particular that the marginal rates of substitutions of goods within one period must agree with the relative spot prices, which is intuitively plausible since there is no additional gains from trade within a period by definition. Furthermore, efficiency implies that the intertemporal marginal rates of substitutions are equated among the household. An important message of Proposition 13 is therefore that in a Quasi- ETEC, the *intertemporal* marginal rate of substitutions, while being equated across agents, *need not coincide with* the respective relative intertemporal market prices. The gap between the common marginal rates of substitution and the respective relative market prices, can be attributed to the implicit transfers that Quasi-ETEC entail. These transfers are indeed what the common distortion parameter β captures.

To see this last point explicitly, normalize the price of the first good in period 0 equal to one, so that now we have $p^{*0} = \bar{p}^0$, and $\beta p^{*1} = \bar{p}^1$ for some β. Since x^* is efficient, by the second fundamental theorem of welfare economics (Lemma 12), there are (i) $\lambda_h > 0$, $h = 1, 2, ..., H$ and a vector $\bar{p} = \left(\bar{p}^0, \bar{p}^1\right) \in \mathbb{R}^L$ such that $\lambda_h \bar{p} = \partial u_h\left(x_h^*\right)$, i.e., both $\lambda_h \bar{p}^0 = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^0}$ and $\lambda_h \bar{p}^1 = \frac{\partial u_h\left(x_h^{0*}, x_h^{1*}\right)}{\partial x_h^1}$ hold for $h = 1, ..., H$, and (ii) for each $h = 1, ..., H$, a real number τ_h such that x^* is a Walrasian equilibrium allocation with transfers when the prices are $\bar{p} = \left(\bar{p}^0, \bar{p}^1\right) \in \mathbb{R}^L$ and the transfers are $(\tau_1, ..., \tau_H)$. Next

$$\tau_h = \bar{p}^0 \cdot \left(x_h^{*0} - e_h^0\right) + \bar{p}^1 \cdot \left(x_h^{*1} - e_h^1\right)$$

holds for each $h = 1, ..., H$. Utilizing $p^{*0} = \bar{p}^0$, and $\beta p^{*1} = \bar{p}^0$, we obtain that

$$\tau_h = p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) + \beta p^{*1} \cdot \left(x_h^{*1} - e_h^1\right)$$

holds for each $h = 1, ..., H$. Next subtracting the left hand side of the equation in *(2)* of the proposition from the RHS of the previous equation, gives that

$$\tau_h = (\beta - 1)\, p^{*1} \cdot \left(x_h^{*1} - e_h^1\right) \tag{9}$$

holds for each $h = 1, ..., H$.

Indeed, if $\beta = 1$, so that there is no distortion, one obtains that the transfers are zero for each household as then the Quasi-ETEC allocation corresponds to an AD equilibrium. If $\beta > 1$, the households who save in period 0 effectively receive a transfer while those who borrow in period 0 are taxed, and vice-versa when $\beta < 1$.

In summary, substituting transfers (9) into the budget, we obtain the following.

Corollary 15 *Let x^* be an efficient allocation. Then $\left(x^*, \left(p^{0*}, p^{1*}\right)\right)$ is a Quasi- ETEC if and only if the following two conditions hold:*
(1) there exists $\beta > 0$ such that for every $h = 1, ..., H$, $\left(x_h^{0*}, x_h^{1*}\right)$ maximizes $u_h\left(x^0, x^1\right)$ subject to $p^{*0} \cdot \left(x_h^0 - x_h^{*0}\right) + \beta p^{*1} \cdot \left(x_h^1 - x_h^{1*}\right) = 0$;
(2) $p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) + p^{*1} \cdot \left(x_h^{*1} - e_h^1\right) = 0$ for each $h = 1, ..., H$.

5 Generic Indeterminacy of Quasi-ETEC

We shall use without proof the following known result about regular economies originated from Debreu (1975) combined with the fundamental theorems (e.g. Lemma 12) which says that efficient allocations and their associated supporting prices can be parametrized by transfers among households[6]:

[6]See Sections 4.4 - 4.7 of Balasko (1988) and sections 4.6 Mas-Colell (1985).

Lemma 16 *There exists a generic set of economies,* $\mathcal{E}_R \subseteq \left(\mathbb{R}^L\right)^H$, *such that for each economy* $\bar{e} \in \mathcal{E}_R$:

(a) there are finitely many PFE;

(b) for each PFE allocation $\bar{x}$ *of economy* $\bar{e}$, *there exists an open set* $V \subseteq \mathcal{E}_R$ *containing* $\bar{e}$ *and a neighborhood* W *of* $0 \in \mathbb{R}^{H-1}$, *and* C^1 *functions* $x_h(w;e) = \left(x_h^0(w;e), x_h^1(w;e)\right)$ *for* $h = 1, ..., H$, *and* $p(w;e) = \left(p^0(w;e), p^1(w;e)\right) >> 0$ *on* $W \times V$ *with the price of the first period 0 good normalized to be one such that*

(i) $\sum_{h=1}^H x_h(w;e) = \sum_h e_h$ *and* $x_h(0;\bar{e}) = \bar{x}_h$ *for all h;*

(ii) if $(\cdots, x_h, \cdots)$ *is a feasible allocation for* $e \in V$ *close enough to* $\bar{x}$, *it is efficient if and only if there is* $w \in W$ *such that* $x_h = x_h(w;e)$ *for all h, and*

(iii) for each $h = 1, ..., H$, $\left(x_h^0(w;e), x_h^1(w;e)\right)$ *maximizes* u_h *subject to*

$$p^0(w;e) \cdot \left(x_h^0 - e_h^0\right) + p^1(w;e) \cdot \left(x_h^1 - e_h^1\right) = w_h, \tag{10}$$

where $w_H = -\sum_{h=1}^{H-1} w_h$. *In particular, a PFE of economy* $e \in V$ *near* $\bar{x}$ *occurs if and only if* $w = 0$.

Applying Lemma 16, starting with an economy $\bar{e} \in \mathcal{E}_R$ and a PFE allocation $\bar{x}$ of $\bar{e}$, find neighborhoods W and V, and C^1 functions x and p. Consider an economy $e \in V$ and its PFE $x^* = x(0;e)$. By construction, efficient allocations around x^* are exactly $\{x(w;e) : w \in W\}$. We ask if $(x_h(w;e))_{h=1}^H$ arises as an Quasi-ETEC allocation of e, i.e., there are $\left(p^{*0}, p^{*1}\right)$ such that $\left((x_h(w;e))_{h=1}^H, \left(p^{*0}, p^{*1}\right)\right)$ is a Quasi-ETEC.

For any given $w \in W$, note that the maximization condition (ii) implies that $\left(p^0(w;e), p^1(w;e)\right)$ is proportional to $\left(\frac{\partial u_h}{\partial x_h^0}\left(x_h^0(w;e), x_h^1(w;e)\right), \frac{\partial u_h}{\partial x_h^1}\left(x_h^0(w;e), x_h^1(w;e)\right)\right)$ for each $h = 1, .., H$, by Lemma 11. Thus from Proposition 13, if $\left((x_h(w;;e))_{h=1}^H, \left(p^{*0}, p^{*1}\right)\right)$ is a Quasi-ETEC, then p^{*0} must be proportional to $p^0(w;e)$, and p^{*1} must be proportional to $p^1(w;e)$. So with the suitable normalization (see Remark 14) we conclude $p^{*0} = p^0(w;e)$, and $\beta p^{*1} = p^1(w;e)$ for some β. That is, an efficient allocation $(x_h(w;e))_{h=1}^H$ is a Quasi-ETEC allocation if and only if there exists $\beta > 0$, such that $\beta p^{1*} = p^1(w)$ and $p^{*0} \cdot \left(x_h^0(w;e) - \bar{e}_h^0\right) + p^{*1} \cdot \left(x_h^1(w;e) - \bar{e}_h^1\right) = p^0(w;e) \cdot \left(x_h^0(w;e) - \bar{e}_h^0\right) + (1/\beta)\, p^1(w;e) \cdot \left(x_h^1(w;e) - \bar{e}_h^1\right) = 0$ hold. Therefore we have shown that $(x_h(w;e))_{h=1}^H$ is a Quasi- ETEC allocation if and only if the following system of equations

$$\beta p^0(w;e) \cdot \left(x_h^0(w;e) - \bar{e}_h^0\right) + p^1(w) \cdot \left(x_h^1(w;e) - \bar{e}_h^1\right) = 0 \text{ for each } h = 1, ..., H, \tag{11}$$

has a solution $\beta > 0$. In view of (10), and keeping in mind that one of the budget equations must be redundant because of the feasibility of the allocation, (11) holds if *and only if* the following system of H equations and $H+1$ variables have a solution:

$$(\beta - 1)\, p^0 (w; e) \cdot \left(x_h^0 (w; e) - \bar{e}_h^0 \right) + w_h = 0, \; h = 1, ..., H - 1 \tag{12}$$
$$\sum_{h=1}^{H} w_h = 0$$

A generic existence and indeterminacy result for Quasi-ETEC can now be established:

Proposition 17 *For any economy $\bar{e} \in \mathcal{E}_R$, there is a neighborhood V of $\bar{e}$ and an interval $\left(\underline{\beta}, \bar{\beta}\right)$ containing 1, and a C^1 function $(x(\beta, e), p(\beta, e))$ defined on $\left(\underline{\beta}, \bar{\beta}\right) \times V$ such that $(x(\beta, e), p(\beta, e))$ is a Quasi- ETEC of $e \in V$. Moreover, if $p^0 (0; e) \cdot \left(x_h^0 (0; e) - e_h^0 \right) \neq 0$ for at least one h, i.e., some households save or borrow at the PFE $x(0; e)$ of e, the set of Quasi-ETEC allocations is a one dimensional manifold around $x(0; e)$.*

Proof. Write x^* for $x(0, e)$ and p^* for $p(0, e)$. Regard the left hand side of (12) as a function $\Phi(w, \beta, e)$. By construction, $\Phi(0, 1, e) = 0$, since the PFE corresponds to $w = 0$ and $\beta = 1$. It then suffices to show that the Jacobian matrix $\frac{\partial}{\partial w \partial \beta} \Phi(0, 1, e)$ has rank H. By direct computation, we find:

$$\frac{\partial}{\partial w \partial \beta} \Phi(0, 1) = \begin{bmatrix} 1 & & 0 & 0 & \vdots \\ & \ddots & & \vdots & p^{*0} \cdot \left(x_h^{*0} - \bar{e}_h^0 \right) \\ 0 & & 1 & 0 & \vdots \\ 1 & \cdots & 1 & 1 & 0 \end{bmatrix},$$

which has rank H since the first H columns are linearly independent.

Applying the implicit function theorem, one can solve w as a function of β around 1 and e and we have

$$\frac{\partial w}{\partial \beta}(1,e) = \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ -1 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) \\ \vdots \\ \sum_{h=1}^{H-1} p^{*0} \cdot \left(x_h^{*0} - e_h^0\right) \end{bmatrix},$$

which is non zero if at least one of $p^{*0} \cdot \left(x_h^{*0} - e_h^0\right)$, $h = 1, ..., H$, is non-zero. Thus under the additional condition about non-trivial savings, the corresponding allocations constitute a one dimensional manifold, parametrized by β around 1. ■

Denote by $\mathcal{E}_R^* \subseteq \mathcal{E}_R$ the set of regular economies where at every equilibrium, some households save or borrow. It can be readily verified that $\mathcal{E}_R^*$ is a generic set, by applying the standard technique of genericity analysis. Proposition 17 says that the set of Quasi- ETEC allocations of economy $e \in \mathcal{E}_R^*$ contains finitely many one dimensional manifolds, as many as the number of PFE, each of which contains one PFE. Since an ETEC allocation must be a Quasi-ETEC allocation, we have the following corollary immediately:

Corollary 18 *For a generic set $\mathcal{E}_R^*$ of economies, the set of ETEC allocations is contained in a one dimensional manifold around a PFE allocation.*

6 Generic Justifiability and Indeterminacy of ETEC

Now we are ready to analyze the structure of ETEC allocations. Recall that a Quasi-ETEC is an ETEC if period 0 consumption bundles are justifiable (Lemma 9). Therefore, given the generic indeterminacy result Proposition 17, the key issue is whether or not a household's consumption bundle close to a PFE is justifiable. Recall that justifiability is not warranted in general (Example 7), and so we seek a generic justifiability result. In principle, justifiability is a property of the individual demand function and it is of independent interest in consumer theory. But for our purpose it suffices to consider consumption vectors around a given PFE consumption bundle, since we are only concerned with Quasi- ETEC allocations with supporting prices which are parametrized by a single parameter, β, as described in Proposition 17.

Pick a regular economy $\bar{e} \in \mathcal{E}_R$ and one of its finitely many PFE equilibrium $(\bar{x}, \bar{p})$, and fix a C^1 parametrization of PFE allocations $\bar{x}(e)$ associated with normalized prices $\bar{p}(e) := \left(\bar{p}^0(e), \bar{p}^1(e)\right)$ defined in a small open set $V \subseteq \mathcal{E}_R$ containing economy $\bar{e}$. We shall show that, if utility functions are time separable,[7] in a generic economy e, for every h, any consumption vector x_h close enough to the PFE consumption $\bar{x}_h$ is justifiable. In fact, the forecast which justify the PFE consumption can be chosen to be proportional to the period 1 PFE prices $\bar{p}^1$.

Let $V(\bar{p};\bar{e}) =: \cap_{h=1}^{H} \{e \in V : \bar{p} \cdot e_h = \bar{p} \cdot \bar{e}_h\} \cap \left\{e \in \mathcal{E} : \sum_{h=1}^{H} e_h = \sum_{h=1}^{H} \bar{e}_h\right\}$. That is, $V(\bar{p};\bar{e})$ is the set of economies with the same total endowments as $\bar{e}$ such that the income level is the same as in PFE $(\bar{x}, \bar{p})$ for all households. The local uniqueness of the PFE assures that $(\bar{x}, \bar{p})$ is also a locally unique PFE of any economy $e \in V(\bar{p};\bar{e})$. Note that by the genericity argument utilizing the vector bundle structure of the equilibrium manifold (see Balasko (1988)), we obtain a desired generic justifiability result if the set of economies where justifiability of the PFE consumption fails is contained in a closed zero measure set in $V(\bar{p};\bar{e})$.

Since there are finitely many households, it is enough to establish this generic property for a fixed household h. Fix a household h from now on, and we shall omit the subscript h when we focus on this particular household to economize notation. We assume a time separable utility function, $u\left(x^0, x^1\right) = W\left(u^0\left(x^0\right), u^1\left(x^1\right)\right)$ for this household. Let the standard competitive demand functions for utility function u^t in period $t = 0, 1$ be $x^t\left(p^t, m^t\right)$ where m^t is the income in period t. Then, an important implication of time separability is that the demand vector for all goods at prices $\left(p^0, p^1\right)$ and income

[7] As we have pointed out earlier, since a time non-separable utility function might induce time inconsistent behavior, the idea of ETEC, and thus the justifiability of a Quasi-ETEC, might be unnecessarily complicated without this assumption.

M is found by solving

$$\max_{m^0,m^1} W\left(u^0\left(x^0\left(p^0,m^0\right)\right),u^1\left(x^1\left(p^1,m^1\right)\right)\right)$$
$$\text{s.t.}$$
$$m^0+m^1=M.$$

Denote by μ^0 and μ^1 the maximizers, which are functions of p^0, p^1, and M. That is, μ^t is the optimal expenditure in period t given prices and the total income. Then the demand vector in period t, $t=0,1$, is $x^t\left(p^t,\mu^t\left(p^0,p^1,M\right)\right)$, i.e., the demand in each period t is just the the demand given period t prices and the optimal expenditure for period t. Under our assumptions, these functions are well defined C^1 functions.

Write $M\left(p^0,p^1\right)$ for the market value of endowments, i.e., $M\left(p^0,p^1\right):=p^0\cdot e^0+p^1\cdot e^1$. Let $I\left(p^0\right):=\left\{\mu^0\left(p^0,t\bar{p}^1,M\left(p^0,t\bar{p}^1\right)\right):t>0\right\}\subset\mathbb{R}$; that is, $I\left(p^0\right)$ is the set of all possible expenditure levels sustained by some forecast which is proportional to $\bar{p}^1$. Set $\underline{\mu}\left(p^0\right):=\inf I\left(p^0\right)$ and $\bar{\mu}\left(p^0\right):=\sup I\left(p^0\right)$. Then we have the following simple sufficient condition for justifiability:

Lemma 19 *Let consumption vector $\tilde{x}^0$ and prices $\tilde{p}^0$ satisfy $\partial u^0\left(\tilde{x}^0\right)=\sigma\tilde{p}^0$ for some $\sigma>0$. Then, there exists a price vector $p^1=t\bar{p}^1$ such that $\tilde{x}^0=x^0\left(\tilde{p}^0,\mu^0\left(\tilde{p}^0,p^1,M\left(\tilde{p}^0,p^1\right)\right)\right)$ if $\underline{\mu}\left(\tilde{p}^0\right)<\tilde{p}^0\cdot\tilde{x}^0<\bar{\mu}\left(\tilde{p}^0\right)$.*

Proof. Let $\tilde{m}=\tilde{p}^0\cdot\tilde{x}^0$. Since $\partial u^0\left(\tilde{x}^0\right)=\sigma\tilde{p}^0$, $\sigma>0$, then from the standard first order condition for utility maximization, it is readily verified that $\tilde{x}^0$ is the demand vector at $\tilde{p}^0$, i.e., $\tilde{x}^0=x^0\left(\tilde{p}^0,\tilde{m}\right)$. If $\underline{\mu}\left(\tilde{p}^0\right)<\tilde{p}^0\cdot\tilde{x}^0<\bar{\mu}\left(\tilde{p}^0\right)$, by the continuity of μ^0 and M, there exists $t>0$ such that $\tilde{m}=\mu^0\left(\tilde{p}^0,t\bar{p}^1,M\left(\tilde{p}^0,t\bar{p}^1\right)\right)$. This price vector $p^1=t\bar{p}^1$ satisfies the desired property. ■

Since $\bar{x}^0$ is the period 0 demand at $\left(\bar{p}^0,\bar{p}^1\right)$ with income $M\left(\bar{p}^0,\bar{p}^1\right)$, it is the demand vector with forecast $\bar{p}^1$, and hence $\underline{\mu}\left(\bar{p}^0\right)\leq\bar{p}^0\cdot\bar{x}^0\leq\bar{\mu}\left(\bar{p}^0\right)$ holds by construction. If the inequalities are strict, i.e., $\underline{\mu}\left(\bar{p}^0\right)<\bar{p}^0\cdot\bar{x}^0<\bar{\mu}\left(\bar{p}^0\right)$, then Lemma 19 applies for a Quasi-ETEC allocation $\tilde{x}^0=x^0\left(\beta;e\right)$ and the corresponding prices $\tilde{p}^0=p^0\left(\beta;e\right)$ which are close to $\left(\bar{x},\bar{p}\right)$. Notice that $\underline{\mu}\left(\bar{p}^0\right)=\bar{p}^0\cdot\bar{x}^0$ occurs only in the unlikely case where the consumption $\bar{x}^0$ corresponds to the minimum period 0 expenditure level on the offer curve. For instance, if there is one good in each period, it means that the PFE consumption occurs exactly at a rare point where the offer curve "bends backward". Similarly, $\bar{\mu}\left(\bar{p}^0\right) = \bar{p}^0 \cdot \bar{x}^0$ appears also unlikely. Our next step is to show that indeed these do not occur generically.

Lemma 20 *Generically in e, $\underline{\mu}\left(\bar{p}^0\right) < \bar{p}^0 \cdot \bar{x}^0 < \bar{\mu}\left(\bar{p}^0\right)$ holds.*

Proof. Write $\xi^t\left(p^0, p^1, M\right) := x^t\left(p^t, \mu^t\left(p^0, p^1, M\right)\right)$, i.e., ξ^t is the standard Walrasian demand function in period t. If $\underline{\mu}\left(\bar{p}^0\right) = \bar{p}^0 \cdot \bar{x}^0$ or $\bar{\mu}\left(\bar{p}^0\right) = \bar{p}^0 \cdot \bar{x}^0$ hold, the period 0 expenditure $\bar{p}^0 \cdot \xi^0\left(\bar{p}^0, t\bar{p}^1, M\left(\bar{p}^0, t\bar{p}^1\right)\right)$ as a function of $t \in (0, \infty)$ is minimized or maximized at $t = 1$. Therefore, the derivative of this function must be zero at $t = 1$. Let $\bar{w} = W\left(u^0\left(\bar{x}^0\right), u^1\left(\bar{x}^1\right)\right)$ and denote the Hicksian demand for period t goods by $\eta^t\left(p^0, p^1, \bar{w}\right)$. Applying the Slutsky decomposition, the first order condition above can be written as

$$\bar{p}^0 \cdot \left(\frac{\partial \eta^0}{\partial p^1} - \frac{\partial \xi^0}{\partial M}\left(\bar{x}^1 - e^1\right)^T\right)\bar{p}^1 = 0, \tag{13}$$

where the derivatives are evaluated at $\left(\bar{p}^0, \bar{p}^1\right)$ and $\bar{w}$.

Claim: $\bar{p}^0 \cdot \frac{\partial \xi^0}{\partial M} \neq 0$. Suppose not, and (13) implies $\bar{p}^0 \cdot \frac{\partial \eta^0}{\partial p^1}\bar{p}^1 = 0$. Since the price vector belongs to the null space of the substitution matrix, we have $\frac{\partial \eta^0}{\partial p^0}\bar{p}^0 + \frac{\partial \eta^0}{\partial p^1}\bar{p}^1 = 0$, and so from $\bar{p}^0 \cdot \frac{\partial \eta^0}{\partial p^1}\bar{p}^1 = 0$ we conclude $\bar{p}^0 \cdot \frac{\partial \eta^0}{\partial p^0}\bar{p}^0 = 0$. But then we would have:

$$\begin{pmatrix} \bar{p}^0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \eta^0}{\partial p^0} & \frac{\partial \eta^0}{\partial p^1} \\ \frac{\partial \eta^1}{\partial p^0} & \frac{\partial \eta^1}{\partial p^1} \end{pmatrix} \begin{pmatrix} \bar{p}^0 \\ 0 \end{pmatrix} = \bar{p}^0 \cdot \frac{\partial \eta^0}{\partial p^0}\bar{p}^0 = 0,$$

which is impossible since the substitution matrix is negative semi-definite and the associated quadratic form assumes value 0 iff the vector is question is proportional to the price vector. It suffices to observe that $\left(\bar{p}^0, 0\right)$ is not proportional to $\left(\bar{p}^0, \bar{p}^1\right) >> 0$. Thus the claim is established.

Recall that as long as $\left(\left(e^0, e^1\right), e_{-h}\right) \in V\left(\bar{p}; \bar{e}\right)$ where e_{-h} denotes the endowments for the other households, the locally unique PFE consumption and prices of the economy remain the same. Since $\bar{p}^0 \cdot \frac{\partial \xi^0}{\partial M} \neq 0$ implies that the left hand side of the first order condition (13) is a non trivial affine function of $\left(e^0, e^1\right)$, and hence if can holds only for a non generic set of economies in $V\left(\bar{p}; \bar{e}\right)$. That is, except for a non generic set of economies in $V\left(\bar{p}; \bar{e}\right)$, $\underline{\mu}\left(\bar{p}^0\right) < \bar{p}^0 \cdot \bar{x}^0 < \bar{\mu}\left(\bar{p}^0\right)$ must hold. Appealing to the aforementioned technique utilizing the vector bundle structure of the equilibrium manifold, we establish the result. ■

With these justifiability results in hand, we are finally ready to state and prove the main result formally.

Proposition 21 *Assume that the utility function is time separable for every household. Then, there exists a generic set of economies $\mathcal{E}^{**}$ such that for each economy $e \in \mathcal{E}^{**}$, (1) there are finitely many PFE; (2) for each PFE allocation, there is a one dimensional set of ETEC allocations containing the PFE allocation.*

Proof. Let $\mathcal{E}^{**} \subset \mathcal{E}_R^*$ be the set of regular economies where at every PFE, every household saves or borrows, and $\underline{\mu}\left(\bar{p}^0\right) < \bar{p}^0 \cdot \bar{x}^0 < \bar{\mu}\left(\tilde{p}^0\right)$ holds. Lemma 20 assures that $\mathcal{E}^{**}$ is a generic set. Condition (1) holds by construction, and condition (2) holds by Proposition 17 and Lemma 19. ■

Recall that under time separability, an ETEC is automatically an ETE. Therefore the result above shows that the set of ETE allocations is also generically one dimensional.

7 Concluding Remarks

7.1 Extension of the main result

Notice that in our analysis of justifiability in Section 6, the desired justifiability is established with a forecast which is proportional to the period 1 PFE prices. Although ETEC does not require any coordination among forecasts, we have in fact shown that the forecasts of households can agree on the relative prices of period 1 goods, and hence the only essential heterogeneity in forecasts across households pertains to their differing forecasts of the rate of inflation in period one.

Of course, this observation heavily depend on the time separability assumption, not to mention Proposition 21 itself. We discuss if our main result Proposition 21 extends beyond the case of time separable utility functions. First of all, without time separability, an ETEC might not be an ETE, and we believe that such an extension can be done only for ETEC once time separability is not warranted. Since the analysis of Quasi-ETEC is done generally, the issue is whether or not a generic one dimensionality of ETEC allocations holds for general utility functions, which is equivalent to ask if an Quasi-ETEC consumption is justifiable.

The analysis of section 6 does not appear to extend easily. For a general utility function, as forecasts vary, the dimension of corresponding consumption vectors would constitute an L^1 dimensional set around the PFE. Thus if $L^0 > L^1$, a simple counting arguments suggest that justifiability is not warranted if the period 0 consumption bundle is given arbitrarily. We therefore believe that a desired extension of Proposition 21, if possible, must take advantage of the structure of equilibrium in a more delicate manner. More generally, it remains to be seen whether generic justifiability can be obtained by allowing utility perturbation in the set of general, non-time separable utility functions.

7.2 Future research

Notice that the one dimensionality of ETEC implies that different forecasts effectively induce, roughly speaking, income transfers up to one dimension between lenders and borrowers. Thus our set up leaves an avenue for policy interventions: a planner (or a central bank) may seek to direct the economy to an appropriate efficient allocation by exercising influence on the forecasts of various households and thereby inducing income transfers. In this context, it appears natural to study ETEC under the postulate that households forecasts agree on the relative prices and disagreements are confined to the rates of inflation, and these estimates of inflation can be influenced by a monetary authority.

Since there is no uncertainty in the model, we only considered a point forecast for households. Since forecasts have no direct welfare implication, it is a reasonable choice to keep the model simple. But we could readily include stochastic forecasts, in the sense that each household might believe period 1 prices are random. This extension has no impact on the analysis for the analysis of Quasi-ETEC. However, such stochastic forecasts increase the set of period 0 consumptions which can be justified. Indeed, an example can be readily constructed to confirm that a consumption vector, not justifiable by a point forecast, might be justifiable with a random forecast. Therefore, the justifiability problem for general utility functions outlined in the previous subsection might be overcome with stochastic forecasts.

We chose the simple two period setup with no uncertainly in order to address the issue of decentralizability of efficient allocations in its purest form. The extensions to models with many periods under uncertainty are interesting and important especially in the context of welfare enhancing policy interventions. For instance, imagine that there are many periods and there is only one good in each period. The analysis of Quasi-ETEC in this paper suggests that the degree of indeterminacy would grow as the number of periods increases, since each period would add an additional route for a bias about inflation. But the relation between Quasi-ETEC and ETE(C) seems more complicated; Quasi-ETEC is mute about the dynamic process of forecasts, whereas there seem to be natural consistency restrictions for dynamic forecasts if there are more than 2 periods.

7.3 Literature

To conclude we briefly discuss literature that accommodates heterogenous forecasts in dynamic models. Kurz (2011) summarizes recent work on the role of diverse market beliefs. The literature on price uncertainty incorporates sometimes incorporates heterogenous forecasts. In particular, there are papers that propose trade in price contingent contracts to deal with the uncertainty; Svensson (1981) considers the case where a complete set of price contingent securities are competitively traded, while Kurz and Wu (1996) draws a connection between rational belief equilibrium, a weakening of rational expectations equilibrium, and a particular notion of Pareto efficiency in a overlapping generations model with complete competitive markets for trading price uncertainty. These models do not address the possibility of obtaining classical Pareto efficiency with heterogenous forecasts in a finite general equilibrium model.

These issues are taken up in Chatterji and Ghosal (2013) and the ETE studied in this paper can be seen as a particular variant of a *perfectly contracted equilibrium* proposed there: in a model of reduced form intertemporal (price-contingent) contracts, a perfectly contracted equilibrium is in effect a Pareto efficient and individually rational allocation which is decentralizable through prices. They showed that a perfectly contracted equilibrium is not necessarily a competitive equilibrium, and moreover, the set of such equilibria contains a set whose dimension is one less than the number of households.

However, the set of intertemporally feasible contracts in their model is unstructured and consequently the meaning of decentralizable contracts is delicate. For instance they do not address issues of retrospective consistency of forecasts or how indeterminacy relates to distortions in interest rates and inflation. On the other hand, we only consider non-price-contingent intertemporal contracts which can arise from explicit decentralized trade in spot markets and a bond market with heterogenous forecasts. Our approach allows us to introduce considerations of retrospective consistency naturally and relate the source of indeterminacy to a common distortion in the effective interest rate.

Earlier work by Chatterji, Kajii and Zeng (2018a, 2018b) established the one dimensional ETE result for the case of economies with one good in each period using a more direct approach which however does not indicate how the results would generalize to the case of multiple goods in each period. While our model is more general since we do allow multiple goods, our principal contribution is the methodology: the notions of retrospective consistency and Quasi ETEC that we propose clearly identify the source and the nature of the indeterminacy in this more general model.

One approach that seeks to explain the prevalence of heterogenous beliefs (or forecasts) uses the notion of eductive stability, a fictitious time coordination procedure based on rationalizability adapted to market settings. Recent work by Guesnerie and Jara-Moroni (2011)) shows using the eductive stability approach that heterogenous beliefs may persist in the simple economic models. It would be interesting to see whether points in the one dimensional set of ETEC allocations that differ from PFE allocations can be obtained as limit points of such coordination procedures on expectations.

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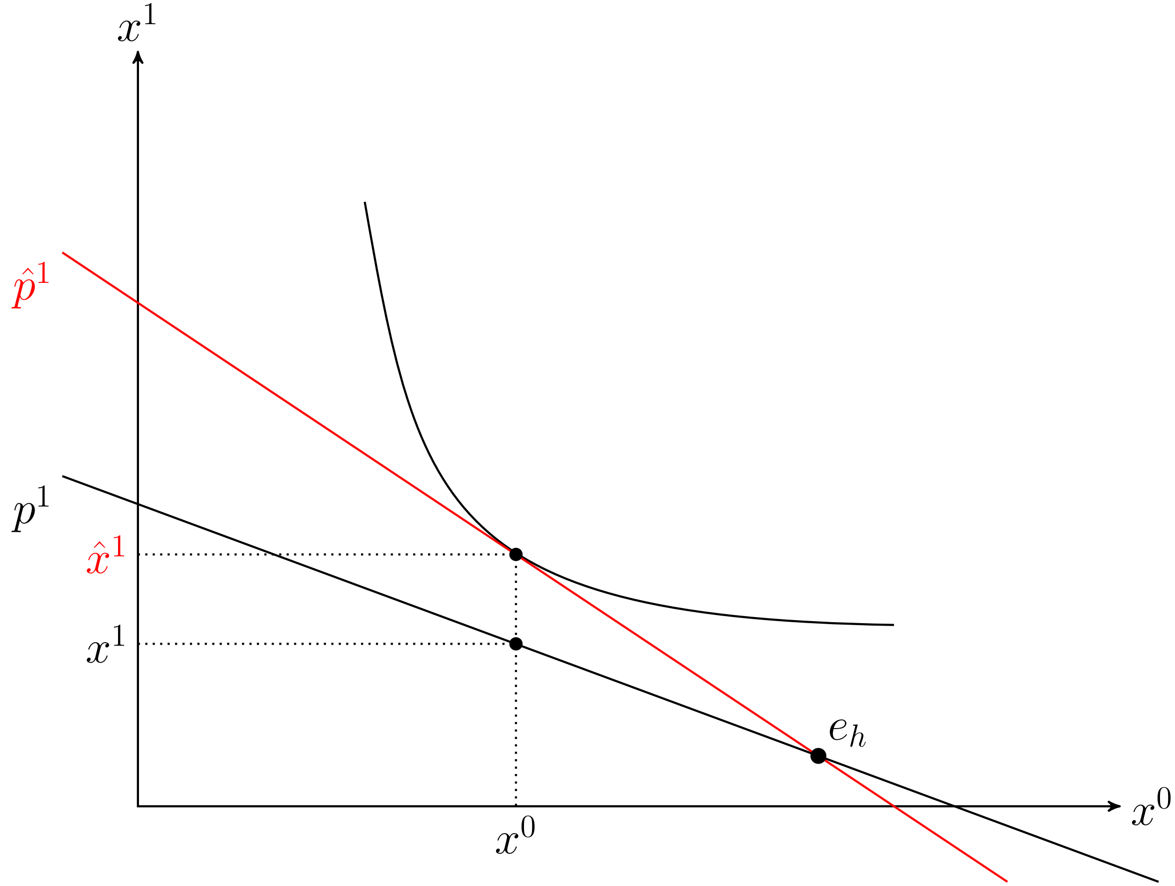

Figure 1

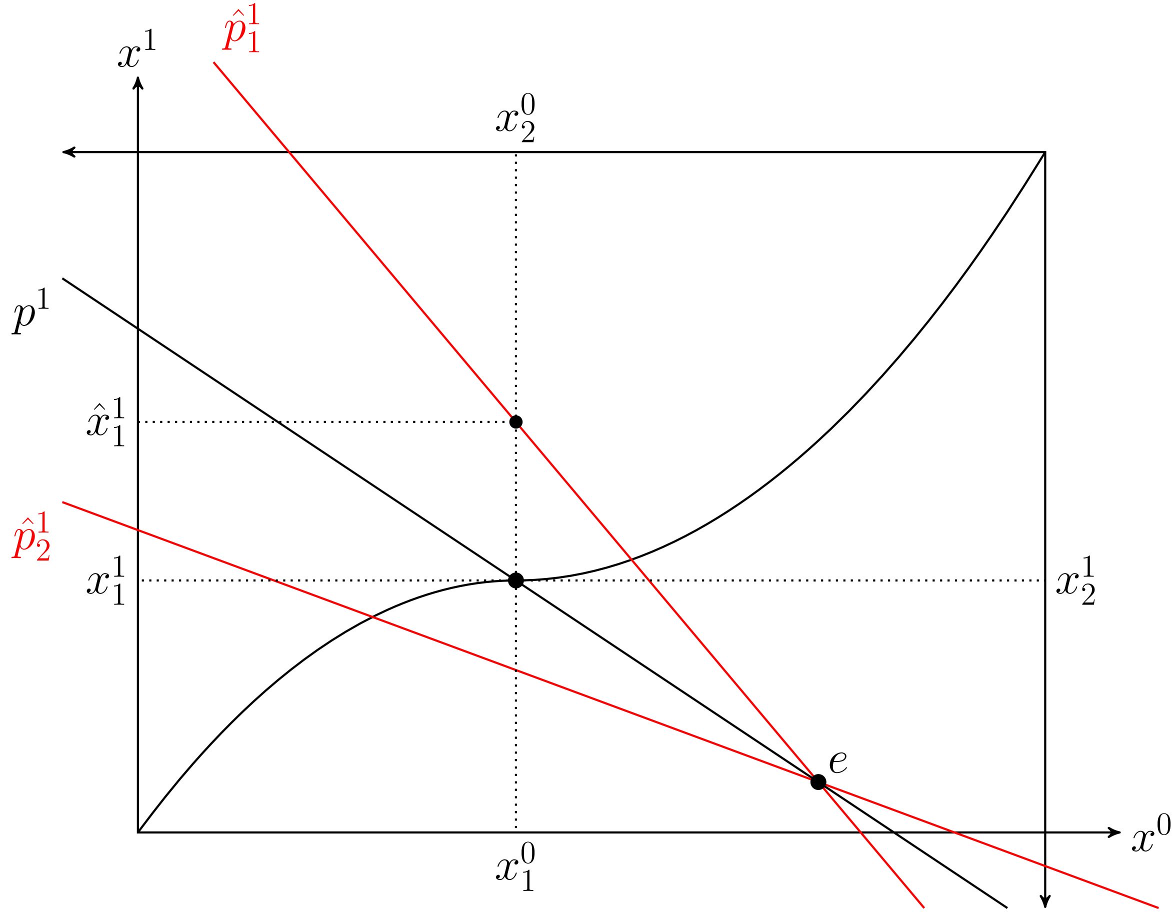

Figure 2