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Notes on "Refinements and Higher Order Beliefs" *

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The abstract of our 1997 survey paper Kajii and Morris (1997b) on "Refinements and Higher Order Beliefs" reads:

> This paper presents a simple framework that allows us to survey and relate some different strands of the game theory literature. We describe a "canonical" way of adding incomplete information to a complete information game. This framework allows us to give a simple "complete theory" interpretation (Kreps 1990) of standard normal form refinements such as perfection, and to relate refinements both to the "higher order beliefs literature" (Rubinstein 1989; Monderer and Samet 1989; Morris, Rob and Shin 1995; Kajii and Morris 1997a) and the "payoff uncertainty approach" (Fudenberg, Kreps and Levine 1988; Dekel and Fudenberg 1990).

In particular, this paper provided a unified framework to relate the notion of *equilibria robust to incomplete information* introduced in Kajii and Morris (1997a) [Hereafter, KM1997] to the classic refinements literature. It followed Fudenberg, Kreps and Levine (1988) and Kreps (1990) in relating refinements of Nash equilibria to a "complete theory" where behavior was rationalized by explicit incomplete information about payoffs, rather than depending on action trembles or other exogenous perturbations. It followed Fudenberg and Tirole (1991), chapter 14, in providing a unified treatment of refinements and a literature on higher-order beliefs rather than proposing a particular solution concept.

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The primary purpose of the survey paper was to promote the idea of robust equilibria in KM1997 and we did not try to publish it as an independent paper. Since we wrote this paper, there have been many developments in the literature on robust equilibria, fortunately. But there has been little work emphasizing a unified perspective, and consequently this paper seems more relevant than ever. We are therefore very happy to publish it twenty years later.[1] We provide some notes in the following on relevant developments in the literature and how they relate to the survey. These notes assume familiarity with the basic concepts introduced in the survey paper and KM1997.

1 Key Developments on Robustness to Incomplete Information

Here are some key developments on the core idea of robustness to incomplete information as a novel refinement:

1.1 Existence

KM1997 provided a sufficient condition for a Nash equilibrium to be robust. Ui (2000) provided a sufficient condition based on the idea of potential games, and Morris and Ui (2005) provided a number of sufficient conditions based on notions of generalized potential games. Since then, a variety of new sufficient conditions for robustness have been developed: examples include Oyama and Tercieux (2009) and Nora and Uno (2014).

KM1997 showed that risk dominance was both necessary and sufficient for robustness to incomplete information in two player two action games. The literature mentioned above has provided stronger sufficient conditions but not necessary conditions. Oyama and Takahashi (2019) provided necessary and sufficient conditions for robustness in many player, two action, asymmetric payoff, supermodular games.

1.2 Continuity

The solution concept of KM1997 imposed the requirement that an incomplete information elaboration of a complete information game was close to a complete information game if, with high probability, players *know* that payoffs are given by the complete information game. If one replaces "know" with "believe with high probability", similar results reported in the paper hold, and there was a somewhat arbitrary modelling choice. In this sense

[1] The survey paper (Kajii and Morris 1997b) will be included in a special issue of Japanese Economic Review to be published in 2020. Rather than re-writing the paper, we have decided to publish it as it was originally written.

the robust equilibrium is continuous with respect to limiting believes for fixed payoffs.

Haimanko and Kajii (2016) showed however that it is not continuous in payoffs, i.e., the solution correspondence which maps the complete information game's payoffs to robust equilibria is not upper hemi continuous at some games. An equilibrium strategy profile in an incomplete information game is an ε-equilibrium of any nearby game, and so demanding an equilibrium in each elaboration was too stringent. Haimanko and Kajii (2016) then showed that if the notion is weakened by considering ε-equilibria in an incomplete information elaboration for arbitrarily small ε, then this weaker robust equilibrium is upper hemicontinuous in payoffs. Carmona (2017) examined robust equilibria in an abstract setting from the perspective of generic continuity of the solution correspondence.

1.3 Set based Robustness

In KM1997, we examined the robustness of singleton Nash equilibria to incomplete information. It is very natural to look at a set based notion of robustness: do there exist a set of correlated equilibria of a complete information game with the property that every incomplete information game where payoffs are almost always given by that complete information have an equilibrium where play is in that set of correlated equilibria? The set of all correlated equilibria automatically satisfy this property, but the interesting question is whether there are interesting sufficient conditions for a small set to be robust in this sense.

Morris and Ui (2005) formally defined a set-based version of robustness and provide a number of sufficient conditions based on notions of generalized potential games.

1.4 Canonical versus Full Elaborations

The canonical elaborations studied in our unified survey were invented for a clean exposition of the role of the *infection argument* and the *critical path result* in KM1997. Because of the additional structure, the set of canonical elaborations is smaller than that of general elaborations considered in KM1997, thus the robustness in the survey is in principle a weaker concept. While this does not matter for the results reported in the survey, the distinction turned out to be important in later work. For instance, Ui (2000) established that potential maximizing Nash equilibria of potential games are robust to canonical elaborations, but not for general elaborations.

Whether or not the robustness to canonical elaborations is equivalent to the robustness in KM1997 in general then remained an open question for many years. Recently, Pram (2018) gave one answer by showing that for the modified correlated equilibrium version of robustness, there is equivalence.

On the other hand, Takahashi (2018) provided an example establishing the failure of equivalence under the original definition, although the counterexample works for the set valued version of robustness.

1.5 Unified Approach

As we wrote earlier, the survey paper was valuable because of its unified framework. However the survey was only about normal form games, whereas the majority of the refinements literature is inspired by extensive form games (and thus non-generic normal form games).

An important recent paper of Takahashi and Tercieux (2018) studied - in the language of our survey - "robustness to all limit common knowledge elaborations" and develops a number of important tight characterizations of the connection between robustness and classical perturbation refinements in the tradition of Kohlberg and Mertens (1986). In particular, they show (roughly, translated into our language) that hyperstability is sufficient for robustness to all limit independent elaborations, that essentiality is sufficient for robustness to all canonical singleton elaborations and that stability is equivalent to robustness to all canonical, independent, singleton, known own payoff elaborations.

2 Related Works

We shall provide notes of somewhat more distantly related works:

1. One can think of a “committed type” in a canonical elaboration as those who are unable to make inference about other players' actions. The type who believes their opponents are committed types can then be interpreted as those who can do one more layer of inference, so they thinks that they are playing only against “committed types”. Murayama (2015) elaborated on this idea to study robustness of predictions when players can only reason to a finite number of levels, relating this questions to the robustness to incomplete information question.

2. Robustness to incomplete information is closely related the global games literature initiated by Carlsson and van Damme (1993). In particular, robustness to incomplete information turns out to be sufficient for being the noise independent selection, in the sense of Frankel, Morris and Pauzner (1993), because, loosely speaking global game perturbations are an intermediate class of sequences, more general that limit common knowledge but with some restrictions beyond robustness. Morris and Shin (2003) described the connection heuristically; some of the same sufficient conditions for noise independent selection and robustness respectively appear in Frankel, Morris and Pauzner

(1993) and Morris and Ui (2005). Formal proofs of versions of the connection appear in the working paper version of Basteck, Daniels and Heinemann (2013) and Oury and Tercieux (2007). Oyama and Takahashi (2011), Basteck and Daniels (2011) and Honda (2011) together identified a gap between robustness to all elaborations and robustness to global game elaborations: in two player, three action supermodular games with symmetric payoffs, there is generically an equilibrium that is the noise independent selection in global games, but it is not in general robust to incomplete information.

3. The survey focussed on common prior information structures. One could examine non-common prior analogues of all the results. Oyama and Tercieux (2010) have examined a non common prior version of robustness. In generic games, only unique strategies surviving iterated deletion of strictly dominated strategies are robust to all elaborations. They also show a smallest robust set exists and is equal to the set of a posteriori equilibria (a refinement of subjective correlated equilibria). There is a close connection between this result and Weinstein and Yildiz (2007) who examined an interim notion of robustness in the universal type space. In this interim version, the common prior assumption no longer has the bite that it has in the framework of KM1997 and Oyama and Tercieux (2010). See also Germano, Weinstein and Zuazo-Garin (2016) and Penta and Zuazo-Garin (2018). Many other results in the survey will also have non-common prior analogues. For example, Frick and Romm (2015) provided an analysis of non-common prior robustness to some limit common knowledge elaboration, showing that it gives the solution concept of $S^{\infty}W$ (one round of deletion of weakly dominated strategies followed by iterated deletion of strictly dominated strategies) of Borgers (1994) and Dekel and Fudenberg (1990).

4. Chassang and Takahashi (2011) considered a repeated game version of robustness in the sense of KM1997.

5. The technique in KM1997 is to use "belief operators" - introduced by Monderer and Samet (1989) - to characterize equilibrium behavior in arbitrary common prior type spaces and then state global results that hold across all common prior type spaces exploiting that characterization. Morris and Shin (2007) and Morris, Shin and Yildiz (2015) generalized these belief operators - to be state dependent and record beliefs about multiple events at once - and Oyama and Takahashi (2019) use such generalized belief operators to generalize the scope of the critical path result from KM1997.

6. Morris (1997) described a formal equivalence between the analysis of

incomplete information games and games played on networks. Under this equivalence, a symmetric action profile being "robust to incomplete information" in the sense of KM1997 translates to that action being uninvadable under best response dynamics in a network game. This is discussed in Morris (1997), lies behind results (but is not emphasized) in Morris (2000) and is developed explicitly in Oyama and Takahashi (2015).

7. The ideas developed in the robustness literature turn out to be very relevant in characterizing the solution to many player information design problems where the information designer anticipates adverserial equilibrium selection: see Bergemann and Morris (2019), section 7.

References

[1] Basteck, Christian and Tijman Daniëls (2011). "Every symmetric 3×3 global game of strategic complementarities has noise independent selection," *Journal of Mathematical Economics* **47**, 749-754.

[2] Basteck, Christian, Tijman Daniels and Frank Heinemann (2013). "Characterizing Equilibrium Selection in Global Games with Strategic Complementarities," *Journal of Economic Theory* **148**, 2620-2637.

[3] Bergemann, Dirk and Stephen Morris (2019). "Information Design: A Unified Perspective," *Journal of Economic Literature* **57**, 1-57 (forthcoming).

[4] Borgers, T. (1994). "Weak Dominance and Approximate Common Knowledge." *Journal of Economic Theory* **64**, 265.

[5] Carlsson, Hans and Eric van Damme (1993). "Global Games and Equilibrium Selection," *Econometrica* **61**, 989-1018.

[6] Carmona, Guilherme (2018) "On the generic robustness of solution concepts to incomplete information," *Journal of Mathematical Economics* **75**, issue C, 13-18.

[7] Chassang, Sylvain and Satoru Takashashi (2011). "Robustness to Incomplete Information in Repeated Games," *Theoretical Economics* **6**, 49-63.

[8] Dekel, Eddie and Drew Fudenberg (1990). "Rational Behavior with Payoff Uncertainty," *Journal of Economic Theory* **52**, 243-267.

[9] Frankel, David, Stephen Morris and Adi Pauzner (2003). "Equilibrium Selection in Global Games with Strategic Complementaries," *Journal of Economic Theory* **108**,1-44.

[10] Frick, Mira and Assaf Romm (2015). "Rational Behavior under Correlated Uncertainty," *Journal of Economic Theory* **160**, 56-71.

[11] Fudenberg, Drew, David Kreps and David Levine (1988). "On the Robustness of Equilibrium Refinements," *Journal of Economic Theory* **44**, 354-380.

[12] Fudenberg, Drew and Jean Tirole (1991). *Game Theory.* M.I.T. Press.

[13] Germano, Fabrizio, Jonathan Weinstein and Peio Zuazo-Garin (2016). "Uncertain Rationality, Depth of Reasoning and Robustness in Games with Incomplete Information."

[14] Haimanko, Ori and Atsushi Kajii (2016). "Approximate Robustness of Equilibrium to Incomplete Information," *International Journal of Game Theory* **45**, 839-857.

[15] Honda, Jun (2011). "Noise-Independent Selection in Global Games and Monotone Potential Maximizer: A Symmetric Example," *Journal of Mathematical Economics* **47**, 663-669.

[16] Kajii, Atsushi and Stephen Morris (1997a). "The Robustness of Equilibria to Incomplete Information," *Econometrica* **65**, 1283-1309.

[17] Kajii, Atsushi and Stephen Morris (1997b). "Refinements and Higher Order Beliefs: A Unified Survey," Northwestern Center Working Paper No. 1197. Available at SSRN: https://ssrn.com/abstract=283477.

[18] Kohlberg, Elon and J.-F. Mertens (1986). "On the Strategic Stability of Equilibria," *Econometrica* **54**, 1003-1037.

[19] Kreps, David (1990). *Game Theory and Economic Modelling.* Oxford: Clarendon Press.

[20] Monderer, Dov and Dov Samet (1989). "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior* **1**, 170-190.

[21] Morris, Stephen (1997). "Interaction Games: A Unified Analysis of Incomplete Information, Local Interaction and Random Matching," https://ssrn.com/abstract=290880.

[22] Morris, Stephen (2000). "Contagion," *Review of Economic Studies* **67**, 57-78.

[23] Morris, Stephen and Hyun Song Shin (2003). "Global Games: Theory and Applications," in *Advances in Economics and Econometrics: Theory and Applications: Proceedings of the 8th World Congress of*

the Econometric Society, volume 1. Cambridge University Press, Cambridge.

[24] Morris, Stephen and Hyun Song Shin (2007). "Common Belief Foundations of Global Games," working paper, Princeton University, https://scholar.princeton.edu/sites/default/files/smorris/files/cbf.pdf.

[25] Morris, Stephen and Hyun Song Shin (2016). "Common Belief Foundations of Global Games," *Journal of Economic Theory* **163**, 826-848.

[26] Morris, Stephen and Takashi Ui (2005). "Generalized Potentials and Robust Sets of Equilibria," *Journal of Economic Theory* **124**, 45-78.

[27] Murayama, Kota (2015). "Robust Predictions under Finite Depth of Reasoning," working paper.

[28] Nora, Vladyslav and Hiroshi Uno (2014). "Saddle Functions and Robust Sets of Equilibria," *Journal of Economic Theory* **150**, 866-877.

[29] Oury, Marion and Olivier Tercieux (2007). "Contagion in Games with Strategic Complementarities," working paper.

[30] Oyama, Daisuke and Satoru Takahashi (2011). "On the Relationship between Robustness to Incomplete Information and Noise-Independent Selection in Global Games," *Journal of Mathematical Economics* **47**, 683-688.

[31] Oyama, Daisuke and Satoru Takahashi (2015). "Contagion and Uninvadability in Local Interaction Games: The Bilingual Game and General Supermodular Games," *Journal of Economic Theory* **157**, 100-127.

[32] Oyama, Daisuke and Satoru Takahashi (2018). "Generalized Belief Operators and the Impact of Small Probability Events" working paper, University of Tokyo and National University of Singapore.

[33] Oyama, Daisuke and Olivier Tercieux (2009). "Iterated Potential and Robustness of Equilibria," *Journal of Economic Theory* **144**, 1726-1769.

[34] Oyama, Daisuke and Olivier Tercieux (2010). "Robust Equilibria under Non-Common Priors," *Journal of Economic Theory* **145**, 752-784.

[35] Penta, Antonio and Peio Zuazo-Garin (2018). "Rationality and Observability," working paper.

[36] Pram, Kyl (2018). "On the Equivalence of Robustness to Canonical and General Elaborations," forthcoming in *Journal of Economic Theory.*

[37] Rob, Rafael Stephen Morris and Hyun Song Shin (1995). "**p**-Dominance and Belief Potential," *Econometrica* **63**, 145-157.

[38] Rubinstein, Ariel. (1989). “The Electronic Mail Game: Strategic Behavior under Almost Common Knowledge,” *American Economic Review* **79**, 385-391.

[39] Takahashi, Satoru (2018). "On the Non-Equivalence of All and Canonical Elaborations," working paper, National University of Singapore.

[40] Takahashi, Satoru and Olivier Tercieux (2018). "Robust Equilibrium Outcomes in Sequential Games under Almost Certainty of Payoffs," working paper.

[41] Ui, Takashi (2001). "Robust Equilibria of Potential Games," *Econometrica* **69**, 1373-1380.

[42] Weinstein, Jonathan and Muhamet Yildiz (2007). "On a Structure Theorem for Rationalizability with application to Robust Predictions of Refinements," *Econometrica* **75**, 365-400.