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Exchange Rates and Fundamentals: Closing a Two-country Model

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Abstract

In an influential paper, Engel and West (2005) claim that the near random-walk behavior of nominal exchange rates is an equilibrium outcome of a variant of present-value models when economic fundamentals follow exogenous first-order integrated processes and the discount factor approaches one. Subsequent empirical studies further confirm this proposition by estimating a discount factor that is close to one under distinct identification schemes. In this paper, I argue that the unit market discount factor implies the counterfactual joint equilibrium dynamics of random-walk exchange rates and economic fundamentals within a canonical, two-country, incomplete market model. Bayesian posterior simulation exercises of a two-country model based on post-Bretton Woods data from Canada and the United States reveal difficulties in reconciling the equilibrium random-walk proposition within the two-country model; in particular, the market discount factor is identified as being much lower than one.

Key Words: Exchange rates; Present-value model; Economic fundamentals; Random walk; Two-country model; Incomplete markets; Cointegrated TFPs; Debt elastic risk premium.

JEL Classification Number: E31, E37, and F41

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1. Introduction

Few equilibrium models for nominal exchange rates systematically beat a naive random-walk counterpart in terms of out-of-sample forecast performance. Since the study of Meese and Rogoff (1983), this robust empirical property of nominal exchange rate fluctuations has stubbornly resisted theoretical challenges to understand the behavior of nominal exchange rates as equilibrium outcomes. The recently developed open-economy dynamic stochastic general equilibrium (DSGE) models also suffer from this problem. Infamous as the disconnect puzzle, open-economy DSGE models fail to generate random-walk nominal exchange rates along an equilibrium path because their exchange rate forecasts are closely related to other macroeconomic fundamentals.

In a recent paper, Engel and West (2005, hereafter EW) establish the near random-walk behavior of nominal exchange rates within a partial equilibrium asset approach. Their model implies that equilibrium nominal exchange rates are given as the present discounted values of the expected future values of economic fundamentals. If economic fundamentals are integrated of order one (hereafter I(1)) and the discount factor approaches one, a nominal exchange rate then follows a near random-walk process in equilibrium. This equilibrium random-walk property is attributable to the fact that only the Beveridge-Nelson trend components in the I(1) economic fundamentals are reflected in present-value calculation at the limit of the unit discount factor. Because the Beveridge-Nelson permanent component is a random walk, the current economic fundamentals lack the power to forecast future depreciation rates even along an equilibrium path.[1]

Because the assumed non-stationarity of economic fundamentals seems to hold without question, subsequent studies within the literature have focused on the empirical validity of the assumption that the discount factor is close to one. Examining data on different currencies and spanning distinct sample periods, Sarno and Sojli (2009) and Balke et al. (2013) identify a discount factor based on partial equilibrium asset approaches seminar to that of EW and infer that the estimated discount factor is indeed distributed near to one.

Nason and Rogers (2008, hereafter NR) attempt to generalize EW's proposition more rigorously and preserve the random-walk property of nominal exchange rates within a two-country dynamic stochastic general equilibrium (DSGE) model that includes incomplete international financial markets. NR rely only on a subset of the first-order necessary conditions (FONCs) of the proposed two-country model, i.e., the utility-based, uncovered interest rate parity (UIP) condition, money demand functions, and purchasing power parity (PPP) condition, to construct the present value model of nominal exchange rates (DSGE-PVM). In their DSGE-PVM, an equilibrium nominal exchange rate is given as the present discounted values of the expected future values of fundamen-

[1] Nominal exchange rates, therefore, need to Granger-cause future economic fundamentals, not vice versa. The empirical exercises of EW based on vector autoregressions (VARs) provide solid evidence for this implication of Granger-causality across different currencies.

tals that consist of cross-country consumption and money supply differentials. As claimed in EW, if these fundamentals are I(1), the nominal exchange rate behaves like a near random-walk at the limit of the unit market discount factor.

Utilizing the cross-equation restrictions (CERs) of the DSGE-PVM and specifying the exogenous I(1) processes of the economic fundamentals, NR estimate a restricted unobserved component (UC) model for the bilateral exchange rate between Canada and the United States. Their Bayesian posterior inferences using post-Bretton Woods data support the hypothesis of EW, finding that the market discount factor is close to one. Moreover, they observe that permanent shocks to the money supply and consumption differentials dominate the historical movements of the bilateral exchange rate.[2]

In this paper, I try to go beyond the theoretical and empirical achievements of NR. My challenge of reconciling random-walk exchange rates within a two-country general equilibrium model begins by arguing that NR somehow stop short before closing their two-country model. There are three areas for concern in their empirical exercise based on the DSGE-PVM. First, NR construct their DSGE-PVM by taking the log-linear approximations of the stochastically de-trended FONCs around a stable, deterministic, steady state. The incompleteness of the international financial market in their two-country model, in which only state non-contingent bonds are traded by representative households across the two countries, might lead endogenous variables to exhibit permanent unit-root dynamics. In this case, there is no guarantee that any stable, deterministic, steady state will exist.[3]

Second, the assumed I(1) consumption differential is inconsistent with the balanced growth path of the two-country model, which is endowed with single consumption goods. The source of the non-stationary consumption differential is the assumption that the cross-country differential in the total factor productivity (TFP) is I(1). In the exercise of NR, each country's endogenous variables are stochastically de-trended with their own countries' TFPs. The de-trended market-clearing condition of consumption goods, which is equivalent to the de-trended resource constraint, depends on the TFP differential. In this case, the non-stationary TFP differential makes the de-trended resource constraint violate the balanced growth restriction.

Finally, the third concern is that NR omit the Euler equations for the optimal intertemporal consumption allocations of both countries and treat the consumption differential as an exogenous random variable. The omitted CERs that the Euler equations impose on the consumption differential, however, might result in the serious misidentification and misevaluation of the two-country

[2]This empirical result is consistent with the argument known as the PPP puzzle (Rogoff 1996) because, by incorporating price stickiness, many open-economy DSGE models emphasize the role of mean-reverting monetary policy shocks as the main force driving nominal exchange rates.

[3]See the detailed discussions of Ghironi (2006) and Boileu and Normandin (2008) regarding the non-stationarity problem inherent to incomplete asset market models.

general equilibrium model as the true data-generating mechanism of random-walk exchange rates. The third concern is primarily relevant once I recognize that each country's consumption is determined by the permanent income hypothesis (PIH) and depends substantially on the I(1) endowment and the unit market discount factor, as does the nominal exchange rate. In fact, to my best knowledge, no past study has taken into consideration the endogeneity of economic fundamentals toward the discount factor. The joint determination of nominal exchange rates and economic fundamentals within a single two-country model, hence, might lead to a statistical inference on the discount factor that is sharply different from those of past studies.

To address these three concerns, I investigate a canonical, single-good, two-country, endowment economy model in which international financial markets are utilized as a device for intertemporal consumption-smoothing. The model used in this paper is quite stylized but similar to that of NR except with regard to two important aspects. The first is that the model contains a debt-elastic risk premium. As characterized by Schmitt-Grohé and Uribe (2003) in a small open-economy model and Boileu and Normandin (2008) in a two-country international business cycle model, a debt-elastic risk premium has served as a popular instrument to induce the stationarity of the equilibrium-balanced growth path.[4] I show that by introducing a wedge between the world and country-specific interest rates, the debt-elastic risk premium alters the UIP condition and makes the resulting present-value model of the nominal exchange rate different from the DSGE-PVM in NR.

The second aspect that differentiates this paper's model from that of NR is that the stochastic trends in both countries appear to be independent in the short run but dependent in the long run. In this model, the exogenous endowment processes of the two countries consist of both permanent and transitory components. I then allow the stochastic trends of the two countries, which are interpreted as their TFPs, to be cointegrated, as emphasized in recent papers by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013) in the context of international business cycles. In this case, because the TFP differential is stationary in population, the equilibrium-balanced growth path is guaranteed to exist. Moreover, if the technological diffusion speed reflected in cointegration is set as sufficiently slow, the TFP differential is empirically identified as an I(1) process with a finite sample. This conjecture is consistent with the empirical finding of NR that a unit root in the cross-country consumption differential cannot be rejected.

Harnessing all the FONCs of the model to endogenously determine the nominal exchange rate and the consumption differential along the unique equilibrium path, I show that the expected equilibrium currency return is characterized by a linear function of the de-trended net foreign

[4] A non-exhaustive list of studies that adopt a debt-elastic risk premium as a device to avoid the non-stationarity problem in open-economy DSGE models includes Nason and Rogers (2006), Adolfson et al.(2007), Kano (2009), Justiniano and Preston(2010), García-Cicco et al. (2010), and Bodenstein (2011).

asset position and other transitory components.[5] When the market discount factor approaches one, this dependence of the expected currency return on the transitory components of the model vanishes asymptotically. Therefore, the near random-walk property of the equilibrium exchange rate indeed holds even after the two-country model is suitably closed. Importantly, the model generates a tractable analytical solution of equilibrium random-walk exchange rates in cases with two symmetric countries. The resulting closed-form solution reveals that the exchange rate is primarily driven by permanent shocks to the money supply differential, among other stationary shocks. This stringent theoretical prediction echoes the findings of NR. However, in contrast to the claim of NR, the permanent but cointegrated TFP shocks cannot be significant drivers of the random-walk nominal exchange rate because the TFP differential should be stationary to close the two-country model.

In addition, the investigation in this paper goes even further. I also characterize the equilibrium consumption differential through an analytical solution. The resulting closed-from representation of relative consumption reveals that at the limit of the unit market discount factor, the consumption differential is perfectly correlated with the PPP deviation, i.e., the real exchange rate (RER). This implication stems from two theoretically crucial facts. First, consumption in each country does not rely on any monetary shocks due to the classical dichotomy of this model, which does not include price stickiness. Second, at the limit of the unit discount factor, no country-specific endowment shock has a significant impact on the present discounted values of expected future endowment differentials because of the balanced growth restriction. The resulting homogeneity of the permanent income calculation across the two countries makes their consumptions, defined as half of the global aggregate endowment, identical. Consequently, neither permanent nor transitory idiosyncratic endowment shock matters for the consumption differential. Only the relative price, i.e., the RER, has an immediate effect on the consumption differential. The resulting perfect correlation between relative consumption and the RER has been recognized as a major empirical difficulty related to a broad class of international business cycle models since that of Backus and Smith (1993).[6]

A macroeconometrician who tries to fit the model to both the exchange rate and consumption fundamental data then faces a serious trade-off. On the one hand, if he or she fits the model to the near random-walk exchange rate, the market discount factor should be close to one. The model, however, tends to impose two unrealistic theoretical restrictions on the data — a permanent money supply differential shock as the dominant driver of random-walk exchange rates and the infamous Backus and Smith problem of an implausibly strong connection between relative consumption and

[5] Such transitory components include transitory money supply differentials, transitory money demand differentials, and transitory PPP deviation shocks.

[6] Because the model used in this paper does not include non-tradable goods, the RER is not determined endogenously in this model, as in the two-country incomplete market model of Benigno and Thoenissen (2008). Rather, the RER is defined as the exogenous shocks to the PPP deviation, as specified in EW and Balke at al. (2013).

the RER. On the other hand, if she or he tries to avoid these counterfactual restrictions by sufficiently lowering the discount factor, the model loses its ability to generate the near random-walk behavior of nominal exchange rates.

An obvious empirical question then is how seriously is the statistical inference on the market discount factor affected by this theoretical trade-off? To address this question, I estimate a statistical UC model that is fully restricted by the proposed two-country model with a Bayesian posterior simulation method. Given relevant prior distributions of the model 's structural parameters, which are especially intended to identify permanent and transitory shocks, the same post-Bretton Woods data for Canada and the United States investigated in NR then finds that the market discount factor is *a posteriori* distributed around 0.62. Notice that this market discount factor size is far below the size close to one that is statistically inferred by many recent empirical studies under different identification strategies. The observation of this paper, hence, implies the theoretical trade-off mentioned above is indeed severe: It is still a quite difficult task to explain data variations in the nominal exchange rate and the corresponding macroeconomic fundamentals jointly and consistently within a canonical, open-economy, general equilibrium framework once such a stylized, two-country, incomplete market model is closed correctly.

The remainder of this paper is organized as follows. In the next section, I introduce the two-country incomplete market model employed in this paper. Section 3 then derives and discusses the equilibrium random-walk property of nominal exchange rates and the Backus and Smith puzzle of a perfect correlation between relative consumption and the RER at the limit of the unit market discount factor. After reporting the main results of the Bayesian exercises in section 4, I conclude in section 5.

2. A two-country incomplete market model

2.1. The model

In this paper, I investigate a canonical incomplete market model with two countries, the home (h) and foreign (f) countries. Each country is endowed with a representative household whose objective is the lifetime money-in-utility

$$\sum_{j=0}^{\infty} \beta^j E_t \left\{ \ln C_{i,t+j} + \phi_{i,t+j} \ln \left(\frac{M_{i,t+j}}{P_{i,t+j}} \right) \right\}, \quad 0 < \beta < 1, \quad \text{for } i = h, f,$$

where $C_{i,t}$, $M_{i,t}$, and $P_{i,t}$ represent the ith country's consumption, money stock, and price index, respectively. The money-in-utility function is subject to a persistent money demand shock $\phi_{i,t}$. The representative households in the home and foreign countries maximize their lifetime utility functions subject to the home budget constraint

$$B^h_{h,t} + S_t B^f_{h,t} + P_{h,t} C_{h,t} + M_{h,t} = (1 + r^h_{h,t-1}) B^h_{h,t-1} + S_t (1 + r^f_{h,t-1}) B^f_{h,t-1} + M_{h,t-1} + P_{h,t} Y_{h,t} + T_{h,t}, \quad (1)$$

and its foreign counterpart

$$\frac{B^h_{f,t}}{S_t} + B^f_{f,t} + P_{f,t}C_{f,t} + M_{f,t} = (1+r^h_{f,t-1})\frac{B^h_{f,t-1}}{S_t} + (1+r^f_{f,t-1})B^f_{f,t-1} + M_{f,t-1} + P_{f,t}Y_{f,t} + T_{f,t}, \quad (2)$$

respectively, where $B^l_{i,t}$, $r^l_{i,t}$, $Y_{i,t}$, $T_{i,t}$, and S_t denote the ith country's holdings of the lth country's nominal bonds at the end of time t, the ith county's returns on the lth country's bonds, the ith country's output level, the ith country's government transfers, and the level of the bilateral nominal exchange rate, respectively. Each country's output $Y_{i,t}$ is given as an exogenous endowment following a stochastic process $Y_{i,t} = y_{i,t}A_{i,t}$, where $y_{i,t}$ is the transitory component and $A_{i,t}$ is the permanent component. Below, I interpret the permanent component $A_{i,t}$ as the TFP in the underlying production technology.

The first-order necessary conditions (FONCs) of the home country's household are given by the budget constraint (1), the Euler equation

$$\frac{1}{P_{h,t}C_{h,t}} = \beta(1+r^h_{h,t})E_t\left(\frac{1}{P_{h,t+1}C_{h,t+1}}\right), \quad (3)$$

the utility-based uncovered parity condition (UIP)

$$(1+r^h_{h,t})E_t\left(\frac{1}{P_{h,t+1}C_{h,t+1}}\right) = \frac{(1+r^f_{h,t})}{S_t}E_t\left(\frac{S_{t+1}}{P_{h,t+1}C_{h,t+1}}\right), \quad (4)$$

and the money demand function

$$\frac{M_{h,t}}{P_{h,t}} = \phi_{h,t}\left(\frac{1+r^h_{h,t}}{r^h_{h,t}}\right)C_{h,t}. \quad (5)$$

The foreign country's FONC counterparts are the budget constraint (2), the Euler equation

$$\frac{1}{P_{f,t}C_{f,t}} = \beta(1+r^f_{f,t})E_t\left(\frac{1}{P_{f,t+1}C_{f,t+1}}\right), \quad (6)$$

the utility-based uncovered parity condition (UIP)

$$(1+r^h_{f,t})E_t\left(\frac{1}{S_{t+1}P_{f,t+1}C_{f,t+1}}\right) = \frac{(1+r^f_{f,t})}{S_t}E_t\left(\frac{1}{P_{f,t+1}C_{f,t+1}}\right), \quad (7)$$

and the money demand function

$$\frac{M_{f,t}}{P_{f,t}} = \phi_{f,t}\left(\frac{1+r^f_{f,t}}{r^f_{f,t}}\right)C_{f,t}. \quad (8)$$

Each country's government transfers the seigniorage to the household as a lump-sum. Hence, the government's budget constraint is

$$M_{i,t} - M_{i,t-1} = T_{i,t}, \quad \text{for } i = h, f.$$

The money supply $M_{i,t}$ is specified to consist of permanent and transitory components, $M^{\tau}_{i,t}$ and $m_{i,t}$: $M_{i,t} \equiv m_{i,t} M^{\tau}_{i,t}$ for $i = h, f$.

To close the model within an incomplete international financial market, I allow for a debt-elastic risk premium in the interest rates faced only by the home country:

$$r^{l}_{h,t} = r^{l}_{w,t}[1 + \psi\{\exp(-B^{l}_{h,t}/M^{\tau}_{l,t} + \bar{d}) - 1\}], \quad \bar{d} \leq 0, \quad \psi > 0, \quad \text{for} \quad l = h, f \tag{9}$$

where $r^{l}_{w,t}$ is the equilibrium world interest rate of the lth country's bond. The risk premium is given as an externality: The household does not take into account the effect of the debt position on the risk premium when maximizing the lifetime utility function. On the other hand, I do not attach a risk premium to the foreign country's interest rates: $r^{l}_{f,t} = r^{l}_{w,t}$ for $l = h, f$.

Following EW and Balke et al. (2013), I assume throughout this paper that purchasing power parity (PPP) holds only up to a persistent PPP deviation shock $\ln q_t$:

$$S_t P_{f,t} = P_{h,t} q_t.$$

The market-clearing conditions of the two countries' bond markets are

$$B^{h}_{h,t} + B^{h}_{f,t} = 0 \quad \text{and} \quad B^{f}_{h,t} + B^{f}_{f,t} = 0,$$

i.e., along an equilibrium path, the world net supply of nominal bonds is zero on a period-by-period basis.

As in NR, I assume that the logarithms of the total factor productivity (TFP) and the permanent component of the money supply, $\ln A_{i,t}$ and $\ln M^{\tau}_{i,t}$, are I(1) for $i = h, f$, and the cross-country differential in the permanent component of money supply, $\ln M^{\tau}_{h,t} - \ln M^{\tau}_{f,t}$, is also I(1):

Assumption 1: $\ln A_{i,t}$ *and* $\ln M^{\tau}_{i,t}$ *are I(1) for* $i = h, f$.

Assumption 2: $\ln M^{\tau}_{h,t} - \ln M^{\tau}_{f,t}$ *is I(1).*

Following Assumptions 1 and 2, I specify each country's monetary growth rate $\Delta \ln M^{\tau}_{i,t}$ to be an independent AR(1) process:

$$\Delta \ln M^{\tau}_{i,t} = (1 - \rho_M) \ln \gamma_M + \rho_M \Delta \ln M^{\tau}_{i,t-1} + \epsilon^{i}_{M,t}, \quad \text{for } i = h, f.$$

where $\ln \gamma_M$ and ρ_M are the mean and AR root, respectively, of the money supply growth rate common to the two countries.

Importantly, I do not make NR's assumption that the cross-country TFP differential, $\ln a_t \equiv \ln A_{h,t} - \ln A_{f,t}$, is I(1). Rather, I assume that the TFP differential is integrated of order zero

(I(0)). This deviation from NR's key assumption stems from the fact that an I(1) TFP differential is inconsistent with the stationarity of the stochastically de-trended model and the deterministic steady state of the resulting equilibrium-balanced growth path, as I will show below in more detail. Notice that Assumption 1 and the stationary TFP differential jointly imply that the TFP of the home country must be cointegrated with that of the foreign country:

Assumption 3: $\ln A_{h,t}$ *and* $\ln A_{f,t}$ *are cointegrated with the cointegrated vector* $[1, -1]$ *and have the error correction models (ECMs)*

$$\begin{aligned}\Delta \ln A_{h,t} &= \ln \gamma_A - \frac{\lambda}{2}(\ln A_{h,t-1} - \ln A_{f,t-1}) + \epsilon^h_{A,t},\\ \Delta \ln A_{f,t} &= \ln \gamma_A + \frac{\lambda}{2}(\ln A_{h,t-1} - \ln A_{f,t-1}) + \epsilon^f_{A,t},\end{aligned} \tag{10}$$

where $\gamma_A > 1$ *is the common drift term and* $\lambda \in [0, 1)$ *is the adjustment speed of the error correction mechanism.*

The cointegration restriction that Assumption 3 imposes on the two countries' TFPs is adopted by recent open-economy DSGE studies by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013). ECMs (10) imply that the cross-country TFP differential is I(0) because

$$\ln a_t = (1 - \lambda) \ln a_{t-1} + \epsilon^h_{A,t} - \epsilon^f_{A,t}.$$

Importantly, if the adjustment speed λ is sufficiently close to zero, the cross-country TFP differential can be realized near I(1), as maintained by NR.

The stochastic process of the logarithm of the transitory output component for each country, $\ln y_{i,t}$, is specified as the following AR(1) process:

$$\ln y_{i,t} = (1 - \rho_y) \ln y_i + \rho_y \ln y_{i,t-1} + \epsilon^i_{y,t},$$

for $i = h, f$. Similarly, the stochastic process of the logarithm of the transitory money supply component for each country, $\ln m_{i,t}$, is specified as the following AR(1) process:

$$\ln m_{i,t} = (1 - \rho_m) \ln m_i + \rho_m \ln m_{i,t-1} + \epsilon^i_{m,t},$$

for $i = h, f$. The three other structural shocks, the home and foreign money demand shocks $\phi_{h,t}$ and $\phi_{f,t}$, respectively, and the PPP shock q_t, follow persistent stationary processes. Specifically, they are characterized by AR(1) processes in terms of the following logarithm:

$$\ln \phi_{i,t} = (1 - \rho_\phi) \ln \phi + \rho_\phi \ln \phi_{i,t-1} + \epsilon^i_{\phi,t},$$

for $i = h, f$ and

$$\ln q_t = \rho_q \ln q_{t-1} + \epsilon_{q,t}.$$

Throughout this paper, I assume that all structural shocks are distributed independently.

2.2. Stochastically de-trended system and log-linear approximation

Define stochastically de-trended variables as $c_{i,t} \equiv C_{i,t}/A_{i,t}$, $p_{i,t} \equiv P_{i,t}A_{i,t}/M^{\tau}_{i,t}$, $b^l_{i,t} \equiv B^l_{i,t}/M^{\tau}_{l,t}$, $\gamma^i_{A,t} \equiv A_{i,t}/A_{i,t-1}$, $\gamma^i_{M,t} \equiv M^{\tau}_{i,t}/M^{\tau}_{i,t-1}$, and $s_t \equiv S_t A_{h,t} M^{\tau}_{f,t}/(A_{f,t}M^{\tau}_{h,t})$. The stochastically de-trended PPP condition is $s_t = p_{h,t}q_t/p_{f,t}$. I can take the stochastic de-trending of the home country's FONCs, (1), (3), (4), (5), and (9), as

$$a_t p_{h,t} c_{h,t} + a_t b^h_{h,t} + s_t b^f_{h,t} = (1 + r^h_{h,t-1}) a_t b^h_{h,t-1}/\gamma^h_{M,t} + (1 + r^f_{h,t-1}) s_t b^f_{h,t-1}/\gamma^f_{M,t} + a_t p_{h,t} y_{h,t}, \quad (11)$$

$$\frac{1}{p_{h,t}c_{h,t}} = \beta(1 + r^h_{h,t})E_t\left(\frac{1}{\gamma^h_{M,t+1}p_{h,t+1}c_{h,t+1}}\right), \quad (12)$$

$$s_t(1 + r^h_{h,t})E_t\left(\frac{1}{p_{h,t+1}c_{h,t+1}\gamma^h_{M,t+1}}\right) = (1 + r^f_{h,t})E_t\left(\frac{s_{t+1}\gamma^f_{A,t+1}}{p_{h,t+1}c_{h,t+1}\gamma^h_{A,t+1}\gamma^f_{M,t+1}}\right), \quad (13)$$

$$\frac{m_{h,t}}{p_{h,t}} = \phi_{h,t}c_{h,t}\left(\frac{1 + r^h_{h,t}}{r^h_{h,t}}\right), \quad (14)$$

$$r^h_{h,t} = r^h_{w,t}[1 + \psi\{\exp(-b^h_{h,t} + \bar{d}) - 1\}], \quad (15)$$

and

$$r^f_{h,t} = r^f_{w,t}[1 + \psi\{\exp(-b^f_{h,t} + \bar{d}) - 1\}]. \quad (16)$$

Similarly, the stochastically de-trended versions of the FONCs of the foreign country, (2) (6), (7), and (8), are

$$q_t p_{h,t} c_{f,t} - s_t b^f_{h,t} - a_t b^h_{h,t} = -(1 + r^f_{w,t-1}) s_t b^f_{h,t-1}/\gamma^f_{M,t} - (1 + r^h_{w,t-1}) a_t b^h_{h,t-1}/\gamma^h_{M,t} + q_t p_{h,t} y_{f,t}, \quad (17)$$

$$\frac{s_t}{q_t p_{h,t} c_{f,t}} = \beta(1 + r^f_{w,t})E_t\frac{s_{t+1}}{\gamma^f_{M,t+1}q_{t+1}p_{h,t+1}c_{f,t+1}}, \quad (18)$$

$$(1 + r^h_{w,t})E_t\left(\frac{\gamma^h_{A,t+1}}{q_{t+1}p_{h,t+1}c_{f,t+1}\gamma^h_{M,t+1}\gamma^f_{A,t+1}}\right) = \frac{(1 + r^f_{w,t})}{s_t}E_t\left(\frac{s_{t+1}}{q_{t+1}p_{h,t+1}c_{f,t+1}\gamma^f_{M,t+1}}\right), \quad (19)$$

and

$$\frac{s_t m_{f,t}}{q_t p_{h,t}} = \phi_{f,t}c_{f,t}\left(\frac{1 + r^f_{w,t}}{r^f_{w,t}}\right). \quad (20)$$

These ten equations, (11) - (20), determine the ten endogenous variables $c_{h,t}$, $c_{f,t}$, $p_{h,t}$, s_t, $b^h_{h,t}$, $b^f_{h,t}$, $r^h_{h,t}$, $r^f_{h,t}$, $r^h_{w,t}$, and $r^f_{w,t}$, given nine exogenous variables $\gamma^h_{M,t}$, $\gamma^f_{M,t}$, $\gamma^h_{A,t}$, $\gamma^f_{A,t}$, a_t, $m_{h,t}$, $m_{f,t}$, $y_{h,t}$, and $y_{f,t}$.[7]

Let $\hat{x}$ denote a percentage deviation of any variable x_t from its deterministic steady state value x^*, $\hat{x} \equiv \ln x_t - \ln x^*$.[8] Also, let $\tilde{x}$ denote a deviation of x from its deterministic steady state, $\tilde{x} = x - x^*$.[9] The log-linear approximation of the stochastically de-trended home budget constraint (11) is

$$\begin{aligned} &p^*_h(c^*_h - y_h)(\hat{p}_{h,t} + \hat{a}_t) + p^*_h c^*_h \hat{c}_{h,t} - p^*_h y_h \hat{y}_{h,t} + \tilde{b}^h_{h,t} + \bar{d}(1-\beta^{-1})(s^*\hat{s}_t + \hat{a}_t) + s^*\tilde{b}^f_{h,t} \\ &\quad = \beta^{-1}\bar{d}[(1+\hat{r}^h_{h,t-1}) - \hat{\gamma}^h_{M,t}] + s^*\beta^{-1}\bar{d}[(1+\hat{r}^f_{h,t-1}) - \hat{\gamma}^f_{M,t}] + \beta^{-1}\tilde{b}^h_{h,t-1} + s^*\beta^{-1}\tilde{b}^f_{h,t-1}; \end{aligned} \tag{21}$$

that of the home Euler equation (12) is

$$\hat{p}_{h,t} + \hat{c}_{h,t} + (1+\hat{r}^h_{h,t}) = E_t(\hat{p}_{h,t+1} + \hat{c}_{h,t+1} + \hat{\gamma}^h_{M,t+1}); \tag{22}$$

that of the home UIP condition (13) is

$$E_t\hat{s}_{t+1} - \hat{s}_t = (1+\hat{r}^h_{h,t}) - (1+\hat{r}^f_{h,t}) + E_t(\hat{\gamma}^h_{A,t+1} - \hat{\gamma}^f_{A,t+1} - \hat{\gamma}^h_{M,t+1} + \hat{\gamma}^f_{M,t+1}); \tag{23}$$

and that of the home money demand function (14) is

$$\hat{p}_{h,t} + \hat{c}_{h,t} - \hat{m}_{h,t} = \frac{1}{r^*}(1+\hat{r}^h_{h,t}) - \hat{\phi}_{h,t}. \tag{24}$$

[7] If the TFP differential a_t is I(1) as assumed in NR, the above system of stochastic difference equations becomes nonstationary through the home and foreign budget constraints (11) and (17) and there is no deterministic steady state to converge. Notice that the cross-country permanent money supply differential $\ln M^\tau_{h,t}/M^\tau_{f,t}$ does not appear in the stochastically de-trended system of the FONCs. In contrast to the TFP differential a_t, the I(1) property of $\ln M^\tau_{h,t}/M^\tau_{f,t}$ in Assumption 2 does not matter for the closing of the model. This might be an obvious result of the model's property that the super-neutrality of money holds in the money-in-utility model: Money growth does not matter for the deterministic steady state.

[8] Notice that at the deterministic steady state, the TFP differential a^* is one. Because of the stationarity of the system of equations (11)-(20), the deterministic steady state is characterized by constants c^*_h, c^*_f, p^*_h, s^*, b^{h*}_h, b^{f*}_h, r^{h*}_h, r^{f*}_h, r^{h*}_w, and r^{f*}_w that satisfy

$$\begin{aligned} b^{h*}_h &= b^{f*}_h = \bar{d}, \\ r^* &\equiv r^{h*}_h = r^{f*}_f = r^{h*}_w = r^{f*}_w = \gamma_M/\beta - 1, \\ s^* &= \frac{y_f(\phi\gamma_M)^{-1}r^* + (y_h+y_f)(1-\beta^{-1})\bar{d}}{y_h(\phi\gamma_M)^{-1}r^* - (y_h+y_f)(1-\beta^{-1})\bar{d}}, \\ p^*_h y_h &= (1-\beta^{-1})(1+s^*)\bar{d} + (\phi\gamma_M)^{-1}r^*, \\ p^*_h c^*_h &= (\phi\gamma_M)^{-1}r^*, \\ c^*_f &= s^* c^*_h. \end{aligned}$$

Below, the steady state value of the nominal market discount factor is denoted by $\kappa \equiv 1/(1+r^*) = \beta/\gamma_M$.

[9] In particular, for an interest rate r_t, $(1+\hat{r}_t) = (r_t - r^*)/(1+r^*)$.

The foreign country's counterparts are the log-linear approximation of the stochastically de-trended foreign budget constraint (17)

$$\begin{aligned}p_h^*(c_f^* - y_f)(\hat{p}_{h,t} + \hat{q}_t) + p_h^* c_f^* \hat{c}_{f,t} - p_h^* y_f \hat{y}_{f,t} - \tilde{b}_{h,t}^h - \bar{d}(1-\beta^{-1})(s^*\hat{s}_t + \hat{a}_t) - s^*\tilde{b}_{h,t}^f \\ = -\beta^{-1}\bar{d}[(1+\hat{r}_{w,t-1}^h) - \hat{\gamma}_{M,t}^h] - s^*\beta^{-1}\bar{d}[(1+\hat{r}_{w,t-1}^f) - \hat{\gamma}_{M,t}^f] - \beta^{-1}\tilde{b}_{h,t-1}^h - s^*\beta^{-1}\tilde{b}_{h,t-1}^f; \quad (25)\end{aligned}$$

that of the foreign Euler equation (18)

$$\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t - (1+\hat{r}_{w,t}^f) = E_t(\hat{s}_{t+1} - \hat{p}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1} - \hat{\gamma}_{M,t+1}^f); \quad (26)$$

that of the foreign UIP condition (19)

$$E_t\hat{s}_{t+1} - \hat{s}_t = (1+\hat{r}_{w,t}^h) - (1+\hat{r}_{w,t}^f) + E_t(\hat{\gamma}_{A,t+1}^h - \hat{\gamma}_{A,t+1}^f - \hat{\gamma}_{M,t+1}^h + \hat{\gamma}_{M,t+1}^f); \quad (27)$$

and that of the home money demand function (14)

$$\hat{s}_t + \hat{m}_{f,t} - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t = -\frac{1}{r^*}(1+\hat{r}_{w,t}^f) + \hat{\phi}_{f,t}. \quad (28)$$

The log-linear approximations of the home country's interest rates (15) and (16) are

$$(1+\hat{r}_{h,t}^h) = (1+\hat{r}_{w,t}^h) - \psi(1-\kappa)\tilde{b}_{h,t}^h, \quad \text{and} \quad (1+\hat{r}_{h,t}^f) = (1+\hat{r}_{w,t}^f) - \psi(1-\kappa)\tilde{b}_{h,t}^f. \quad (29)$$

Notice that the home interest rates (29) redefine the home UIP condition (23) as

$$\begin{aligned}E_t\hat{s}_{t+1} - \hat{s}_t = (1+\hat{r}_{w,t}^h) - (1+\hat{r}_{w,t}^f) - \psi(1-\kappa)(\tilde{b}_{h,t}^h - \tilde{b}_{h,t}^f) \\ + E_t(\hat{\gamma}_{A,t+1}^h - \hat{\gamma}_{A,t+1}^f - \hat{\gamma}_{M,t+1}^h + \hat{\gamma}_{M,t+1}^f).\end{aligned}$$

Comparing the above home UIP condition with the foreign UIP condition (27) implies that the home and foreign bonds are perfectly substitutable along the equilibrium path. Hence, the equilibrium condition $\tilde{b}_t \equiv \tilde{b}_{h,t}^h = \tilde{b}_{h,t}^f$ holds.[10]

3. Random-walk exchange rates and Backus and Smith's anomaly

3.1. Equilibrium random-walk property of nominal exchange rates

I will now show that the equilibrium random-walk property of the exchange rate holds in this two-country model. To prove this proposition, I first derive the DSGE-PVM of the exchange rate as an equilibrium condition. Let c_t, m_t, and ϕ_t denote the de-trended consumption ratio, the transitory money supply ratio, and the money demand shock ratio between the two countries,

[10] Appendix A characterizes the equilibrium transitory dynamics of the model for a simplified case including two symmetric countries.

$c_t \equiv c_{h,t}/c_{f,t}$, $m_t \equiv m_{h,t}/m_{f,t}$, and $\phi_t \equiv \phi_{h,t}/\phi_{f,t}$, respectively. Furthermore, let M_t^τ denote the ratio of the permanent money supplies of the home and foreign countries $M_{h,t}^\tau/M_{f,t}^\tau$; let M_t foreign money supplies of the home to the foreign countries $M_{h,t}/M_{f,t} \equiv m_t M_t^\tau$; and let C_t denote the ratio of the consumptions of the home and foreign countries $C_{h,t}/C_{f,t}$. The home and foreign money demand functions, (24) and (28), and the home interest rates (29) yield the following interest rate differential:

$$(1+\hat{r}_{w,t}^h)-(1+\hat{r}_{w,t}^f)=r^*(\hat{s}_t+\hat{c}_t-\hat{m}_t+\hat{\phi}_t-\hat{q}_t)+\psi(1-\kappa)\tilde{b}_t. \tag{30}$$

Substituting the interest rate differential (30) into the foreign UIP condition (27) leads to the expectational difference equation of the de-trended exchange rate $\hat{s}_t$:

$$\begin{aligned}\hat{s}_t=\kappa E_t\hat{s}_{t+1}-(1-\kappa)\hat{c}_t+(1-\kappa)(\hat{m}_t-\hat{\phi}_t+\hat{q}_t)&\\ -\kappa E_t(\hat{\gamma}_{A,t+1}^h-\hat{\gamma}_{A,t+1}^f-\hat{\gamma}_{M,t+1}^h+\hat{\gamma}_{M,t+1}^f)&-\psi\kappa(1-\kappa)\tilde{b}_t.\end{aligned}$$

After unwinding stochastic trends, the above expectational difference equation can be rewritten as

$$\ln S_t=\kappa E_t\ln S_{t+1}+(1-\kappa)\ln M_t-(1-\kappa)\ln C_t-(1-\kappa)(\ln\phi_t-\ln q_t)-\psi\kappa(1-\kappa)\tilde{b}_t.$$

Solving this expectational difference equation by forward iterations under a suitable transversality condition provides the DSGE-PVM of this model:

$$\ln S_t=(1-\kappa)\sum_{j=0}^{\infty}\kappa^j E_t\left(\ln M_{t+j}-\ln C_{t+j}-\psi\kappa\tilde{b}_{t+j}-\ln\phi_{t+j}+\ln q_{t+j}\right). \tag{31}$$

If the fundamental $\ln M_t - \ln C_t$ is I(1), so is the exchange rate. Moreover, the exchange rate should be cointegrated with the fundamentals. To signify this property, the DSGE-PVM (31) can be rewritten as

$$\begin{aligned}\ln S_t-\ln M_t+\ln C_t=&\sum_{j=1}^{\infty}\kappa^j E_t\left(\Delta\ln M_{t+j}-\Delta\ln C_{t+j}\right)\\ &-(1-\kappa)\sum_{j=0}^{\infty}\kappa^j E_t\left(\psi\kappa\tilde{b}_{t+j}+\ln\phi_{t+j}-\ln q_{t+j}\right).\end{aligned} \tag{32}$$

Since the RHS of eq.(32) is I(0), the exchange rate $\ln S_t$ and the I(1) fundamental $\ln M_t - \ln C_t$ are cointegrated. NR hypothesize the cointegration relation among $\ln S_t$, $\ln M_t$, and $\ln C_t$ based on their DSGE-PVM. The model in this paper theoretically restricts the stationarity of the consumption differential $\ln C_t$ because of Assumption 3 due to the requirement of closing the two-country model.[11]

EW and NR, however, reject the cointegration relation between the exchange rate and fundamentals in actual data for major currencies. In particular, EW suggest other unobservable

[11] If the adjustment speed of the error correction mechanism of both countries' TFPs, λ, is sufficiently slow, the maintained stationarity of the consumption differential is unlikely to be detected with a finite sample.

I(1) components that the standard asset approach does not identify as primary reasons for the failure of the cointegration hypothesis (32). Notice that in the DSGE-PVM (31), the equilibrium exchange rate also depends on the present discounted values of expected future de-trended net foreign asset positions $\tilde{b}_t$, the relative money demand shock $\ln \phi_t$, and the PPP shock $\ln q_t$. As shown in Appendix A in a case including symmetric countries, the stationarity of the de-trended international bond holding $\tilde{b}_t$ relies on the sizes of the debt elasticity of the risk premium ψ as well as the market discount factor κ: if either ψ is sufficiently close to zero or κ approaches one, $\tilde{b}_t$ follows a near-I(1) process. Moreover, as stated by Balke et al. (2013), the relative money demand shock and the PPP shock could be unobservable near I(1) components. The theoretical result of (32) can be interpreted as the empirical failure of cointegration among the exchange rate and economic fundamentals, which is consistent with EW's proposal.

NR claim that the DSGE-PVM (31) implies an error-correction representation of the currency return $\Delta \ln S_t$, in which $\Delta \ln S_t$ depends on the lagged error correction term $\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1}$. Their argument also holds even in this model. Appendix B shows that after rearranging the DSGE-PVM (31) in several steps and using the cointegration relation (32), the currency return is

$$\Delta \ln S_t = \frac{1-\kappa}{\kappa}\left(\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} + \ln \phi_{t-1} - \ln q_{t-1}\right) + \psi(1-\kappa)\tilde{b}_{t-1} + u_{s,t}, \tag{33}$$

where $u_{s,t}$ is the i.i.d., rational expectations error

$$u_{s,t} = (1-\kappa)\sum_{j=0}^{\infty}\kappa^j (E_t - E_{t-1})(\ln M_{t+j} - \ln C_{t+j} - \psi\kappa\tilde{b}_{t+j} + \ln q_{t+j} - \ln \phi_{t+j}).$$

Recall that the DSGE-PVM (31) is constructed as an equilibrium condition from some of the model's FONCs. The general equilibrium property of the model, however, imposes another restriction on the present value of the future fundamentals in the DSGE-PVM (31). Note that combining the log-linearized Euler equations of the home and foreign countries, (22) and (26), with those of the home country's interest rates (29), yields the first-order expectational difference equation of $\ln S_t - \ln M_t + \ln C_t - \ln q_t$:

$$\begin{aligned}\ln S_t - \ln M_t + \ln C_t - \ln q_t = \kappa E_t(&\ln S_{t+1} - \ln M_{t+1} + \ln C_{t+1} - \ln q_{t+1}) \\ &+ \kappa\rho_M\hat{\gamma}_{M,t} + \kappa(\rho_m - 1)\ln m_t - (1-\kappa)\ln \phi_t,\end{aligned}$$

where $\hat{\gamma}_{M,t} \equiv \hat{\gamma}^h_{M,t} - \hat{\gamma}^f_{M,t}$ is the money supply growth rate differential. Because κ is less than one, the difference equation above has the unique forward solution

$$\ln S_t = \ln M_t - \ln C_t + \ln q_t + \frac{\kappa\rho_M}{1-\kappa\rho_M}\hat{\gamma}_{M,t} - \frac{\kappa(1-\rho_m)}{1-\kappa\rho_m}\ln m_t - \frac{1-\kappa}{1-\kappa\rho_\phi}\ln \phi_t \tag{34}$$

under a suitable transversality condition.

Imposing the CER (34) on the error-correction process (33) provides the equilibrium currency return

$$\Delta \ln S_t = \psi(1-\kappa)\tilde{b}_{t-1} + \frac{(1-\kappa)\rho_M}{1-\kappa\rho_M}\hat{\gamma}_{M,t-1} + \frac{(1-\kappa)(1-\rho_\phi)}{1-\kappa\rho_\phi}\ln\phi_{t-1} - \frac{(1-\kappa)(1-\rho_m)}{1-\kappa\rho_m}\ln m_{t-1} + u_{s,t}. \quad (35)$$

Equation (35) clearly shows that any dependence of the currency return on past information emerges through the persistence of the net foreign asset position, the money supply growth differential, the transitory money demand shock differential, and the transitory money supply differential.

The important implication of the equilibrium currency return equation (35) is that the logarithm of the exchange rate follows a Martingale difference sequence at the limit of $\kappa \to 1$ because

$$\lim_{\kappa\to 1} E_t \Delta \ln S_{t+1} = 0.$$

Therefore, in this paper, the exchange rate behaves like a random walk when the market discount factor approaches one along the equilibrium path of the two-country model. The equilibrium currency return equation (35) exhibits no dependence of the currency return on past information in this case. Hence, the equilibrium random walk property of the exchange rate, as found in EW and NR, is also preserved in this model.[12]

In the limiting case with the unit market discount factor, the equilibrium currency return is dominated by the i.i.d. rational expectations error $u_{s,t}$. An advantage of working with a structural two-country model is that the rational expectations error $u_{s,t}$ is now fully interpretable as a linear combination of structural shocks. To see this, note that the rational expectations error $u_{s,t}$ in equilibrium is represented by

$$u_{s,t} = (E_t - E_{t-1})\Delta \ln S_t = \epsilon_{M,t} - \epsilon_{A,t} + (E_t - E_{t-1})\hat{s}_t,$$

where $\epsilon_{M,t} \equiv \epsilon^h_{M,t} - \epsilon^f_{M,t}$ and $\epsilon_{A,t} \equiv \epsilon^h_{A,t} - \epsilon^f_{A,t}$ denote the relative permanent money supply shock and the relative TFP shock, respectively. It is not straightforward, however, to calculate the equilibrium surprise of the de-trended exchange rate $(E_t - E_{t-1})\hat{s}_t$. Appendix A shows that in the special case of two symmetric countries, assuming $\bar{d} = 0$ and $y_h = y_f$, the equilibrium de-trended exchange rate is determined by a linear function of $\tilde{b}_{t-1}$, $\ln a_t$, $\ln m_t$, $\ln \phi_t$, $\ln y_t$, $\ln q_t$, and $\hat{\gamma}_{M,t}$:

$$\hat{s}_t = \frac{\beta\eta - 1}{\beta p_h^* y^*}\tilde{b}_{t-1} + \frac{\beta\eta\lambda}{1-\beta\eta(1-\lambda)}\ln a_t + \frac{1-\kappa}{1-\kappa\rho_m}\ln m_t - \frac{1-\kappa}{1-\kappa\rho_\phi}\ln\phi_t - \frac{1-\beta\eta}{1-\beta\eta\rho_y}\ln y_t + \frac{1-\beta\eta}{1-\beta\eta\rho_q}\ln q_t + \frac{\kappa\rho_M}{1-\kappa\rho_M}\hat{\gamma}_{M,t}, \quad (36)$$

[12] A caveat of the above result is that in this model, κ is given as a function of structural parameters β and γ_M: $\kappa = \beta/\gamma_M$. If $\gamma_M > 1$, as found in the postwar data on money growth rates in Canada and the United States, the admissible range of β between zero and one implies that κ is strictly less than one. In this paper, I assume that the limit of $\kappa \to 1$ is well approximated by the limit of $\beta \to 1$ because γ_M takes a value that is very close to one.

where the constant η, which is less than one, approaches one at the limit of $\kappa \to 1$.[13] Hence, the surprise in the de-trended exchange rate between times t and $t-1$ is

$$(E_t - E_{t-1})\hat{s}_t = \frac{\beta\eta\lambda}{1-\beta\eta(1-\lambda)}\epsilon_{A,t} + \frac{1-\kappa}{1-\kappa\rho_m}\epsilon_{m,t} - \frac{1-\kappa}{1-\kappa\rho_\phi}\epsilon_{\phi,t} - \frac{1-\beta\eta}{1-\beta\eta\rho_y}\epsilon_{y,t} + \frac{1-\beta\eta}{1-\beta\eta\rho_q}\epsilon_{q,t} + \frac{\kappa\rho_M}{1-\kappa\rho_M}\epsilon_{M,t}.$$

where $\epsilon_{m,t} \equiv \epsilon^h_{m,t} - \epsilon^f_{m,t}$, $\epsilon_{\phi,t} \equiv (\epsilon^h_{\phi,t} - \epsilon^f_{\phi,t})$, and $\epsilon_{y,t} \equiv \epsilon^h_{y,t} - \epsilon^f_{y,t}$ denote the relative transitory money supply, the relative transitory money demand, and the relative transitory income shocks. The rational expectations error is then given as an explicit linear function of the structural shocks:

$$u_{s,t} = \frac{1}{1-\kappa\rho_M}\epsilon_{M,t} - \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)}\epsilon_{A,t} + \frac{1-\kappa}{1-\kappa\rho_m}\epsilon_{m,t} - \frac{1-\kappa}{1-\kappa\rho_\phi}\epsilon_{\phi,t} - \frac{1-\beta\eta}{1-\beta\eta\rho_y}\epsilon_{y,t} + \frac{1-\beta\eta}{1-\beta\eta\rho_q}\epsilon_{q,t}.$$

Notice that at the limit of $\kappa \to 1$, the model also implies the subjective discount factor $\beta \to 1$ under a positive deterministic money supply growth rate, $\gamma_M > 1$, which is close to one. In this limiting case, observe that the permanent monetary shock $\epsilon_{M,t}$ surely dominates the rational expectations error $u_{s,t}$ and, as a result, the random walk of the exchange rate.

$$\lim_{\kappa\to 1} \Delta \ln S_t = \lim_{\kappa,\beta,\eta\to 1} u_{s,t} = \frac{1}{1-\rho_M}\epsilon_{M,t}.$$

Therefore, no transitory shock matters for the total variations in the random-walk exchange rate. This is because when $\kappa \to 1$, or equivalently, $r^* \to 0$, the interest rate differential (30) becomes insensitive to the transitory money supply, money demand, and consumption differentials. Hence, the exchange rate turns out to be neutral to any transitory monetary and real shocks.

In contrast to the empirical result of NR, which depends on a more flexible reduced-form specification of the consumption differential, no permanent TFP shock can be a primary driver of the random-walk exchange rate. This result is due to the cointegration of the two-country TFPs: No discrepancy between the two countries' TFPs can be permanent in order to guarantee the equilibrium-balanced growth path. The model's theoretical implication of the dominant role of the permanent money supply shock on the random-walk exchange rate, hence, is too restrictive to trace out the actual data variations in the bilateral nominal exchange rate, at least between Canada and the United States.

3.2. Backus and Smith's puzzle at the limit

[13] As defined in Appendix A, the constant η is one of the two roots of the expectational difference equation of the de-trended net foreign asset position $\tilde{b}_t$. A simple calculation shows that the equilibrium currency return (35) can be derived directly from the CER (36) once the approximated relation $\hat{s}_t \approx \ln S_t + \ln A_t - \ln M^\tau_t$ is recognized.

This model, moreover, has another unrealistic implication on the consumption differential equilibrium dynamics $\ln C_t$ when the discount factor approaches one. At the limit of the unit discount factor, Backus and Smith's (1983) problem of a perfect correlation between relative consumption and the RER emerges even under incomplete international financial markets. To observe this property, taking the first difference of the CER (34) yields

$$\Delta \ln C_t = -\Delta \ln S_t + \frac{(1-\kappa)\rho_M}{1-\kappa\rho_M}\hat{\gamma}_{M,t-1} + \frac{1}{1-\kappa\rho_M}\epsilon_{M,t} + \frac{1-\kappa}{1-\kappa\rho_m}\Delta \ln m_t - \frac{1-\kappa}{1-\kappa\rho_\phi}\Delta \ln \phi_t + \Delta \ln q_t.$$

Substituting the equilibrium currency return (35) into the above equation and exploiting the rational expectations error $u_{s,t}$ leads to the following consumption differential dynamics:

$$\Delta \ln C_t = \Delta \ln q_t - \psi(1-\kappa)\tilde{b}_{t-1} + \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)}\epsilon_{A,t} + \frac{1-\beta\eta}{1-\beta\eta\rho_y}\epsilon_{y,t} - \frac{1-\beta\eta}{1-\beta\eta\rho_q}\epsilon_{q,t}. \tag{37}$$

Notice, therefore, that except through the net foreign asset position, no monetary shock directly matters for the change in the equilibrium consumption differential: As in the standard international business cycle model, only real shocks to the endowments and the PPP deviation affect the equilibrium consumption allocation between the two countries.

Taking the limit of equation (37) above with respect to κ results in

$$\lim_{\kappa,\beta,\eta\to 1} \Delta \ln C_t = \Delta \ln q_t.$$

Thus, relative consumption becomes unrelated to any shocks to the endowments of the two countries but is rather perfectly correlated with the exogenous RER. The intuition behind this result is quite straightforward. In this incomplete market model with the PIH households, consumption in each country is determined by splitting the global aggregate endowment across both countries in each period. The portion of the global aggregate endowment allocated to one country is simply given as the present discounted values of the expected future relative endowments of this country to the other. Because the endowment differential is stationary due to the balanced growth restriction, the unit discount factor at the limit makes the portion converge to a constant; in particular, one-half in the case of two symmetric countries. Consumption in both countries, hence, responds to any endowment shocks in the same fashion. As the result, with the discount factor being close to one, relative consumption depends neither on permanent nor transitory endowment shocks. The only shock that can affect the relative consumption is in the corresponding relative price, i.e., the RER.[14]

[14] More precisely, from Appendix A, the consumption logarithms of the home and foreign countries in terms of the home currency can be solved as

$$2\ln C_{h,t} = \ln Y_{h,t} + \ln q_t Y_{f,t} + \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)}\ln a_t + \frac{1-\beta\eta}{1-\beta\eta\rho_y}\ln y_t - \frac{1-\beta\eta}{1-\beta\eta\rho_q}\ln q_t + \frac{1-\beta\eta}{\beta p_h^* y^*}\tilde{b}_{t-1},$$

4. A Bayesian unobserved component approach

This section empirically explores the question of how significantly the tension emerging at the limit of the unit market discount factor among the three theoretical implications — the random-walk exchange rate, the dominance of permanent money supply differential shocks in the variations in the random-walk exchange rate, and the perfect correlation between the relative consumption and the RER — affects posterior inferences in relation to the market discount factor. For this specific purpose, I simplify the estimation exercise as much as possible by adopting the symmetric version of the two-country model, in which the same structural parameters are shared by both countries. This paper then takes a Bayesian UC approach to the proposed structural two-country model.

4.1. The restricted UC model and posterior simulation strategy

Under the symmetric case with $\bar{d} = 0$, FONCs (21)-(28) are degenerated to the following three expectational difference equations with respect to the three endogenous variables $\hat{s}_t$, $\hat{c}_t$, and $\tilde{b}_t$, given the six exogenous variables $\hat{\gamma}_{M,t}$, $\hat{m}_t$, $\hat{a}_t$, $\hat{y}_t$, $\hat{\phi}_t$, and $\hat{q}_t$:

$$\begin{aligned}
&\hat{s}_t = \kappa E_t \hat{s}_{t+1} - (1-\kappa)\hat{c}_t + (1-\kappa)(\hat{m}_t - \hat{\phi}_t + \hat{q}_t) + \lambda\kappa\hat{a}_t + \kappa E_t \hat{\gamma}_{M,t+1} - \psi\kappa(1-\kappa)\tilde{b}_t, \\
&\hat{s}_t + \hat{c}_t - \hat{q}_t = \kappa E_t(\hat{s}_{t+1} + \hat{c}_{t+1} - \hat{q}_{t+1}) + (1-\kappa)(\hat{m}_t - \hat{\phi}_t) + \kappa E_t \hat{\gamma}_{M,t+1}, \\
&\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + p_h^* y^* (\hat{y}_t - \hat{c}_t), \qquad (38)
\end{aligned}$$

where $y^* = y/4$ and $y = y_h = y_f$. Let $\mathbf{X}_t$ denote an unobserved state vector defined as

$$\mathbf{X}_t = [\hat{s}_t \ \hat{c}_t \ E_t\hat{s}_t \ E_t\hat{c}_t \ \tilde{b}_t \ \hat{\gamma}_{M,t} \ \hat{a}_t \ \hat{m}_t \ \hat{y}_t \ \hat{q}_t \ \hat{\phi}_t]'.$$

Furthermore, let ϵ_t and ω_t denote random vectors consisting of structural shocks and rational expectations errors: $\epsilon_t \equiv [\epsilon_{M,t} \ \epsilon_{A,t} \ \epsilon_{m,t} \ \epsilon_{y,t} \ \epsilon_{q,t} \ \epsilon_{\phi,t}]'$ and $\omega_t \equiv [\hat{s}_t - E_{t-1}\hat{s}_t \ \hat{c}_t - E_{t-1}\hat{c}_t]'$, respectively. In particular, for empirical investigation purposes, I presume that the structural shock vector ϵ_t

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$$2\ln q_t C_{f,t} = \ln Y_{h,t} + \ln q_t Y_{f,t} - \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)}\ln a_t - \frac{1-\beta\eta}{1-\beta\eta\rho_y}\ln y_t + \frac{1-\beta\eta}{1-\beta\eta\rho_q}\ln q_t - \frac{1-\beta\eta}{\beta p_h^* y^*}\tilde{b}_{t-1}.$$

Each country's consumption depends on the log-linearized global aggregate endowment $\ln Y_{h,t} + \ln q_t Y_{f,t}$, the log-linearized country-specific portion of the aggregate endowment $\frac{1-\beta\eta}{[1-\beta\eta(1-\lambda)]}\ln a_t + \frac{1-\beta\eta}{(1-\beta\eta\rho_y)}\ln y_t - \frac{1-\beta\eta}{(1-\beta\eta\rho_q)}\ln q_t$, and the wealth effect of the net foreign asset position $\frac{1-\beta\eta}{\beta p_h^* y^*}\tilde{b}_{t-1}$. If the discount factor approaches one, both the log-linearized country-specific portion and the wealth effect of the net foreign asset position disappear and the log consumption levels become

$$\ln C_{h,t} = \frac{1}{2}(\ln Y_{h,t} + \ln Y_{f,t}) + \frac{1}{2}\ln q_t, \quad \ln C_{f,t} = \frac{1}{2}(\ln Y_{h,t} + \ln Y_{f,t}) - \frac{1}{2}\ln q_t.$$

Relative consumption then turns out to be correlated perfectly with the RER because

$$\ln C_{h,t} - \ln C_{f,t} = \ln q_t.$$

is normally distributed, with a mean of zero and a diagonal variance-covariance matrix Σ: $\epsilon_t \sim i.i.d.N(\mathbf{0}, \Sigma)$ with $diag(\Sigma) = [\sigma_M^2\ \sigma_A^2\ \sigma_m^2\ \sigma_y^2\ \sigma_q^2\ \sigma_\phi^2]'$.

Accompanied by the stochastic processes of the exogenous forcing variables, the linear rational expectations model (38) then implies that

$$\Gamma_0 \mathbf{X}_t = \Gamma_1 \mathbf{X}_{t-1} + \Phi_0 \omega_t + \Phi_1 \epsilon_t,$$

where Γ_0, Γ_1, Φ_0, and Φ_1 are the corresponding coefficient matrices. Applying Sims's (2001) QZ algorithm to the linear rational expectations model above yields a unique solution as the following stationary transition equation of the unobservable state vector:

$$\mathbf{X}_t = \mathbf{F}\mathbf{X}_{t-1} + \Phi \epsilon_t, \tag{39}$$

where $\mathbf{F}$ and Φ are confirmable coefficient matrices.

To construct this paper's UC model, I further expand the unobservable state vector $\mathbf{X}_t$ by the permanent money supply differential $\ln M_t^\tau$ to obtain the augmented state vector $\mathbf{Z}_t$: $\mathbf{Z}_t \equiv [\mathbf{X}_t'\ \ln M_t^\tau]'$. The stochastic process of $\ln M_t^\tau$ and the state transition (39) then imply the following non-stationary transition of the expanded state vector $\mathbf{Z}_t$:

$$\mathbf{Z}_t = \mathbf{G}\mathbf{Z}_{t-1} + \Psi \epsilon_t, \quad \epsilon_t \sim i.i.d.N(\mathbf{0}, \Sigma), \tag{40}$$

where $\mathbf{G}$ and Ψ are confirmable coefficient matrices.

In this paper, I explore time-series data on the log of the consumption differential $\ln C_t$, the log of the output differential $\ln Y_t$, the log of the money supply differential $\ln M_t$, the interest rate differential $r_t \equiv r_{h,t}^h - r_{f,t}^f$, and the log of the bilateral exchange rate $\ln S_t$. Let $\mathbf{Y}_t$ denote the information set that consists of these five time series: $\mathbf{Y}_t \equiv [\ln C_t\ \ln Y_t\ \ln M_t\ r_t\ \ln S_t]'$. It is then straightforward to show that the information set $\mathbf{Y}_t$ is linearly related to the unobservable state vector $\mathbf{Z}_t$ as

$$\mathbf{Y}_t = \mathbf{H}\mathbf{Z}_t, \tag{41}$$

where $\mathbf{H}$ is a confirmable coefficient matrix. The transition equation, the unobserved state (40), and the observation equation (41) jointly consist of a non-stationary state-space representation of the two-country model, which is the restricted UC model estimated in this paper.[15]

Given the data set $\mathbf{Y}^T \equiv \{\mathbf{Y}_t\}_{t=0}^T$, applying the Kalman filter to the UC model provides model likelihood $\mathbf{L}(\mathbf{Y}^T|\theta)$, where θ is the structural parameter vector of the two-country model. Multiplying the likelihood by a prior probability of the structural parameters, $\mathbf{p}(\theta)$, is proportional

[15] The state-space form of the model, (40) and (41), decomposes the I(1) difference-stationary information set $\mathbf{Y}_t$ into permanent and transitory components exploiting the theoretical restrictions provided by the two-country model. Recursion of the Kalman filter for a non-stationary state-space model is explained in detail by Hamilton (1994).

to the corresponding posterior distribution $\mathbf{p}(\theta|\mathbf{Y^T}) \propto \mathbf{p}(\theta)\mathbf{L}(\mathbf{Y}^T|\theta)$ through the Bayes law. The posterior distribution $\mathbf{p}(\theta|\mathbf{Y}^T)$ is simulated by the random-walk Metropolis-Hastings algorithm, as implemented by Schorfheide (2000), Bouakez and Kano (2006), and Kano (2009).

4.2. Data and prior construction

The two countries that I empirically examine in this paper are Canada and the United States as the model's home and foreign countries, respectively. I examine post-Bretton Woods quarterly data for these two countries because they satisfy the model assumptions. The data span the period from Q1:1973 to Q4:2007. All the data included in the information set $\mathbf{Y}^T$, except nominal exchange rates, are seasonally adjusted annual rates.[16]

Table 1 reports the prior distributions of the structural parameters of the two-country model, $\mathbf{p}(\theta)$. Since the main goal of this paper's empirical investigation is to draw a posterior inference on the market discount factor $\kappa \equiv \beta/\gamma_M$, I elicit a uniform prior distribution of κ and let the data tell the posterior position of κ given the identification of the restricted UC model. In so doing, on the one hand, the prior distribution of the mean gross monetary growth rate, γ_M, is intended to tightly cover its sample counterparts in both countries through the Gamma distribution, with a mean of 1.015 and standard deviation of 0.005.[17] On the other hand, the prior distribution of the subjective discount factor β is uniformly distributed between zero and one. As a result, the prior distribution of the market discount factor κ is well approximated as the uniform distribution spread over the support of the unit interval.

To guarantee the stationarity of the de-trended net foreign asset position $\tilde{b}_t$, the debt elasticity of the home risk premium ψ should be positive. I therefore set the prior distribution of ψ to the Gamma distribution, with a mean of 0.010 and standard deviation of 0.001. Closing the model also requires the technological diffusion speed λ to be positive but less than one. This necessary condition for the equilibrium-balanced growth path elicits the prior distribution of λ as the Beta distribution, with a mean of 0.010 and standard deviation of 0.001. The slow technological diffusion that the prior mean of λ implies is intended to capture the slow-moving time-series properties observed in the actual consumption and output differentials between Canada and the United States. The prior distribution of the mean monetary demand shock ϕ follows the Gamma distribution, with a mean of 1.000 and small standard deviation of 0.010. By doing so, I assume *a priori* that the monetary demand shock has no effect on the deterministic steady state.

I admit a small persistence of the permanent money growth rate by setting the prior distribution of the AR(1) coefficient ρ_M to the Beta distribution, with a mean of 0.100 and standard deviation of 0.010. The PPP deviation shock, i.e., the RER shock, is presumed to be very persis-

[16] Appendix C provides a detailed description of the source and construction of the data examined in this paper.

[17] The sample mean of the M1 money supply's gross growth rate is 1.016 for Canada and 1.014 for the United States.

tent, as observed by many past empirical studies on the RER. The AR(1) coefficient of the RER, ρ_q, is then accompanied by the Beta prior distribution, with a mean of 0.850 and standard deviation of 0.050. This prior distribution mimics fairly well the posterior distribution of the same structural parameter reported in Figure 3 of Balke et al. (2013), who used a long annual sample of data from the United Kingdom and the United States.[18] On the other hand, there is no robust empirical consensus on the extent of the persistence of the money demand shock. Hence I allow the prior distribution of the AR(1) coefficient of the money demand shock, ρ_ϕ, to be distributed around 0.900 following the Beta distribution, with a mean of 0.900 and a large standard distribution of 0.100. The resulting 95 % coverage, indeed, is [0.633 0.998], which also covers the corresponding posterior distribution displayed within Figure 3 of Balke et al. (2013). Furthermore, to better identify the permanent components of the money supplies and TFPs of both countries, I assume that the corresponding transitory components are white noise by setting the prior mass points of the AR(1) coefficients ρ_m and ρ_y to zero. Finally, the prior standard deviations of all the structural shocks are assumed to share the identical inverse-Gamma distribution, with a mean of 0.010 and standard deviation of 0.010. This prior distribution of Σ yields a higher marginal likelihood among small perturbations. Below, I refer to this prior specification as the Benchmark model.

4.3. Main Results

The second, third, and fourth columns of Table 2 describe the posterior distributions of the structural parameters under the Benchmark model. The most striking posterior inference conveyed by these columns is that the market discount factor κ is identified as being far below one. As displayed in the first row, the data pin down the location of κ very tightly around the posterior mean of 0.612, with a standard deviation of 0.002. This posterior distribution of the market discount factor is too low to guarantee the second necessary condition of the equilibrium random-walk exchange rate established by EW and NR, i.e., that the market discount factor is sufficiently close to one. The other significant result in Table 2 relates to the posterior inferences on the money demand differential shock, ρ_ϕ and σ_ϕ: The data show a more persistent and volatile money demand differential shock compared to the prior specification of the Benchmark model. Notice that the posterior mean of ρ_ϕ is 0.996 and almost 10 % larger than its prior mean value; the posterior mean of σ_ϕ is 0.027 and 17 % larger than its prior mean value. The very persistent money demand differential shock provides evidence that such a structural shock could play a significant role in actual exchange rate movements.

Does this lower market discount factor deteriorate the model's fit to actual exchange rate movements? The answer is clearly no, although the equilibrium currency return depends slightly on past economic fundamentals. The estimated Benchmark model is indeed successful in explaining

[18] In fact, the 90 % interval of [0.739 0.933] includes the most inferences on RER persistence established in major past studies (see, e.g., Rogoff 1996 and Lothian and Taylor 2000).

the historical trajectory of the exchange rate. Figure 1 plots the actual depreciation rate of the Canadian dollar against the United States dollar as the solid black line. The same figure also displays the sum of the Kalman smoothers of the impact effects of both the permanent money supply differential shock $\epsilon_{M,t}$ and the money demand differential shock $\epsilon_{\phi,t}$ for the Benchmark model (the dashed blue line).[19] Although several sharp deviations from the actual data are observed, the impact effects of the two monetary shocks $\epsilon_{M,t}$ and $\epsilon_{\phi,t}$ jointly track the actual depreciation rate of the Canadian dollar fairly well, even under such a low market discount factor.

More generally, each window in Figure 2 corresponding to a particular structural shock exhibits the Kalman smoothed impact effect of the corresponding shock on the actual depreciation rate; the upper-left window corresponds to the TFP differential shock $\epsilon_{A,t}$; the upper-middle window corresponds to the transitory money supply differential shock $\epsilon_{m,t}$; the upper-right window corresponds to the transitory output differential shock $\epsilon_{y,t}$; the lower-left window corresponds to the PPP deviation shock $\epsilon_{q,t}$; the lower-center window corresponds to the money demand differential shock $\epsilon_{\phi,t}$; and, finally, the lower-right window corresponds to the permanent money supply differential shock $\epsilon_{M,t}$. The six windows in Figure 2 reveal that the most important structural shock for the near-random-walk exchange rate between Canada and the United States is identified as the very persistent money demand differential shock in conjunction with the permanent money supply differential shock. This result strongly echoes the finding of Balke et al. (2013), which was based on a different identification in which the persistent money demand differential shock dominates the historical movements of the exchange rate between the United Kingdom and the United States. Nevertheless, it is important to note that the estimated low discount factor allows the TFP differential and the transitory money supply differential shocks to contribute to actual exchange rate fluctuations to some non-negligible degree. In particular, the impact effect of the TFP differential shock appears to be significant, although much smaller than those of the permanent money supply and the money demand shocks. This result is in sharp contrast to NR's inference that the permanent TFP differential is a major driver of the Canada-United States exchange rate.

The Benchmark model also does an excellent job of explaining actual data variations in the output, consumption, and money supply differentials. Figure 3 demonstrates this impressive property of the Benchmark model. The left window in the figure plots the actual log output differential as the solid black line and the Kalman smoother of the TFP differential $\ln A_t$ as the dashed blue line. The window clearly shows that the smoothed TFP differential explains almost all the variations in the actual output differential. The right window of this figure indicates the close fit of the Kalman smoother of the permanent money supply differential $\ln M_t^\tau$ (the dashed blue line) to the actual money supply differential (the solid black line). Finally, the middle window of the figure

[19]The Kalman smoothers are evaluated at the posterior means of the structural parameters. The impact effects of the structural shocks on the depreciation rate are calculated as the last row of the matrix $\mathbf{H}\Psi\epsilon_{t|T}$, where $\epsilon_{t|T}$ is the Kalman smoother of the structural shock ϵ_t.

then depicts the actual log consumption differential (the solid black line) and the Kalman smoother of the TFP differential $\ln A_t$ (the dashed blue line). Notice that the smoothed TFP differential tracks the slow-moving trend component of the actual consumption differential quite successfully, although there are significant deviations of the smoothed inference from its actual data counterpart, especially at the beginning and end of the sample. The structural shock that dominantly generates the actual consumption differential, hence, is identified as the TFP differential shock.

The dominant role that the TFP differential shock plays in the actual consumption differential is also confirmed in Figure 4. In this figure, each window corresponding to a particular structural shock exhibits the Kalman smoothed impact effect of the corresponding shock on the consumption differential change (the dashed blue line) along with the actual consumption differential change (the solid black line). Observe, in the upper-left window, that the smoothed TFP differential shock almost matches its actual-data counterpart. The other structural shocks are unlikely to have any significant effect on the variations in the consumption differential at all.

4.4. Understanding lower discount factors: The High Discount Factor model

Why does the Benchmark model result in such a lower discount factor? To understand this question, I conduct an alternative Bayesian posterior simulation exercise. In this exercise, I intend to fix the discount factor close to one and observe how the empirical performance of the model changes relative to that of the Benchmark model. In so doing, I replace the uniform prior distribution of β in the Benchmark model with a more informative Beta distribution, with a mean of 0.999 and standard deviation of 0.001, and stay with the same prior distributions of the remainder of the structural parameters as in the Benchmark model. I refer to this new specification as the High Discount Factor (HDF) model.

The fifth, sixth, and seventh columns of Table 2 correspond to the posterior distributions of the structural parameters under the HDF model. Observe that the resulting posterior distributions of both the market and subjective discount factors are much closer to one, with posterior means of 0.926 and 0.998, respectively. Crucial changes in the posterior distributions of the structural parameters from the Benchmark model, then, are recognized as significant increases in the posterior means of the standard deviations of the three monetary shocks, σ_M, σ_m, and σ_ϕ. The HDF model, therefore, requires greater volatilities in all the monetary shocks to explain the data.

An important difference between the Benchmark and HDF models is related to the overall fit to the data as well as the Kalman smoothed inferences. First, the last row of Table 2 reports the estimated marginal likelihood for each model.[20] The HDF model yields a smaller marginal likelihood

[20]This paper estimates the marginal likelihoods by using Geweke's (1999) modified harmonic mean estimator. A marginal likelihood is the probability of data $\mathbf{Y}^T$ conditional on an underlying model. In general, the higher the marginal likelihood is, the better the underlying model's overall fit to the data.

of 1787.058 compared to that of the Benchmark model, which was 2145.267. The difference in the marginal likelihoods of the two models is so significant that I conclude that forcing the discount factor to be close to one makes the HDF model's overall fit to the data much worse than that of the Benchmark model. Figure 5 more clearly reveals the source of this significant deterioration of the HDF model compared to the Benchmark model with respect to the marginal likelihood. This figure plots the one-period-ahead forecast errors of the Benchmark and HDF models toward the actual data as the dashed blue and dotted red lines, respectively.[21] The figure clearly shows the greatest difficulty for the HDF model relative to the Benchmark model relates to its fit to the money supply differential.

Furthermore, the dotted red and dashed blue lines displayed in Figures 1, 2, 3, and 4 indicate the HDF and the Benchmark model's Kalman smoothers, respectively. These smoothed inferences convey three properties of the HDF model: (i) the permanent money supply differential shock and the persistent money demand differential shock jointly and dominantly explain actual exchange rate movements; (ii) the PPP deviation shock, not the TFP shock as in the Benchmark model, is the dominant driver of the consumption differential; and (iii) the permanent money supply differential shock fails to track the actual money supply differential as sharply as in the Benchmark model. The first property echoes the main finding of the Benchmark model. The second property, however, represents the drawback of the HDF model. As seen in section 3, with a high discount factor, the consumption differential almost perfectly matches the exogenous PPP deviation shock, of which the exchange rate becomes independent. In the HDF model, the PPP deviation shock, hence, acts as a free latent variable to dominantly explain the consumption differential. This second property, however, is counterfactual and known as the Backus and Smith puzzle.[22] The third property again confirms my conjecture from the low marginal likelihood and the forecast errors of the HDF model relative to the Benchmark model.

Why does the HDF model fail to explain the money supply differential? Remember the model's implication at the limit of the unit discount factor: in contrast to the Benchmark model, with a lower discount factor, the exchange rate data should be explained exclusively by either the permanent money supply differential shock or the very persistent money demand differential shock or both. No TFP differential shock has any effect on the exchange rate in this case. The permanent money supply differential shock, however, needs to track the actual trajectory of the money supply differential as well. These two restrictions on the permanent money supply differential shock then force (i) the persistent money demand differential shock to play a dominant role in explaining the exchange rate and (ii) the white noise transitory money supply differential shock to act as a significant driver of the money supply differential. Because the second requirement implies that an

[21]The forecast errors of the two models are calculated through the Kalman filter forward recursion.

[22]This result implies that if I expand the information set $\mathbf{Y}^T$ by including the PPP data, the HDF model further loses overall data fit.

i.i.d. noise is not negligible in the money supply differential, the in-sample fit of the HDF model to the money supply differential should deteriorate.

5. Conclusions

In this paper, I try to reconcile the random-walk property of nominal exchange rates with a canonical two-country endowment model including incomplete international financial markets. The main challenge undertaken in this paper is to establish the joint equilibrium dynamics of nominal exchange rates and economic fundamentals, both of which should be endogenously determined by the two-country model. After closing the model correctly by allowing the TFPs of both countries to be cointegrated, I discover the equilibrium random walk property of exchange rates when the cross-country money supply differential contains a permanent component and the market discount factor approaches one. The assumption for the equilibrium random-walk exchange rate that the discount factor is close to one, however, imposes unrealistic restrictions on the data — permanent money supply differential shocks as the dominant driver of random-walk exchange rates and the Backus and Smith puzzle of a perfect correlation between relative consumption and the RER.

Bayesian posterior simulation exercises based on post-Bretton Woods data from Canada and the United States reveal a major difficulty in reconciling the random-walk exchange rate and the economic fundamentals with the proposed two-country model. Indeed, under the benchmark identification of the model, the data change the value of the market discount factor to far below one. A more detailed investigation of the model in which the market discount factor is set sufficiently high *a priori* empirically confirms the theoretical conjecture that the posterior inference of a low market discount factor stems from the fact that the model suffers from the Backus and Smith puzzle and that it fails to explain the actual money supply differential.

This paper's findings of such a low discount factor are in sharp contrast to those of high market discount factors in past empirical studies such as NR, Sarno and Sojli (2009), and Balke at al. (2013). Because these past studies did not jointly consider the endogenous determination of economic fundamentals with nominal exchange rates, the general equilibrium consideration sought by this paper is relevant to the better understanding of the near random-walk behavior of nominal exchange rates within structural open-economy models. Identifying an open economy DSGE model that can reconcile the joint equilibrium dynamics of random-walk exchange rates and economic fundamentals under an empirically plausible market discount factor value is a serious open question to be addressed. Since the most crucial difference between the empirical exercise in this paper and that of NR's is in the differing stochastic treatments of the TFP differential and, as a result, the consumption differential, it would be a promising research direction to search for a model-consistent way of allowing the TFP differential to be I(1) without violating the balanced growth restriction.

I leave this challenging question as a valuable for future studies on open-economy macroeconomics to undertake.

Appendix A. Solving a case with two symmetric countries

To understand the equilibrium transitory dynamics of the exchange rate in this model, it is informative to scrutinize a simpler version of the model that includes two symmetric countries. For this purpose, I set the parameter $\bar{d}$ to zero and assume that the transitory output components of the two countries, y_h and y_f, are equal to y. Notice that the deterministic steady state in this case is characterized by $s^* = 1$, $c_h^* = c_f^* = y$, and $p_h^* = (\phi\gamma_M)^{-1} r^*$, where $r^* = \gamma_M/\beta - 1$.

I combine the log-linearized Euler equations of the home and foreign countries, (22) and (26), with those of the home country's interest rates (29) to yield the first-order expectational difference equation of $\hat{s}_t + \hat{c}_{h,t} - \hat{c}_{f,t}$:

$$\hat{s}_t + \hat{c}_{h,t} - \hat{c}_{f,t} - \hat{q}_t = \kappa E_t(\hat{s}_{t+1} + \hat{c}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1}) + \kappa E_t \hat{\gamma}_{M,t+1} + (1-\kappa)(\hat{m}_t - \hat{\phi}_t).$$

Since κ takes a value between zero and one, the above expectational difference equation has a forward solution of $\hat{s}_t + \hat{c}_{h,t} - \hat{c}_{f,t} - \hat{q}_t = \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} + (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t - (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$ under a suitable transversality condition. By, exploiting this forward solution and the stochastic processes of both countries' TFPs (10), I rewrite the foreign UIP condition (27) as

$$E_t\hat{s}_{t+1} - \hat{s}_t = \psi(1-\kappa)\tilde{b}_t - \lambda\hat{a}_t - \frac{\kappa\rho_M(1-\rho_M)}{1-\kappa\rho_M}\hat{\gamma}_{M,t} - \frac{(1-\kappa)(1-\rho_m)}{1-\kappa\rho_m}\hat{m}_t + \frac{(1-\kappa)(1-\rho_\phi)}{1-\kappa\rho_\phi}\hat{\phi}_t, \qquad \text{(A.1)}$$

Furthermore, taking a difference between the log-linearized budget constraints of the home and foreign countries, (21) and (25), I find the law of motion of the international bond holdings

$$\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + p_h^* y^* \hat{s}_t - p_h^* y^* \hat{q}_t - \frac{p_h^* y^* \kappa\rho_M}{1-\kappa\rho_M}\hat{\gamma}_{M,t} - \frac{p_h^* y^*(1-\kappa)}{1-\kappa\rho_m}\hat{m}_t + \frac{p_h^* y^*(1-\kappa)}{1-\kappa\rho_\phi}\hat{\phi}_t + p_h^* y^* \hat{y}_t, \qquad \text{(A.2)}$$

where $y^* = y/4$ and $\hat{y}_t \equiv \hat{y}_{h,t} - \hat{y}_{f,t}$.

Combining equation (A.1) with equation (A.2) then yields the following second-order expectational difference equation with respect to international bond holdings:

$$E_t\tilde{b}_{t+1} - [1 + \beta^{-1} + p_h^* y^* \psi(1-\kappa)]\tilde{b}_t + \beta^{-1}\tilde{b}_{t-1} = -\lambda p_h^* y^* \hat{a}_t + p_h^* y^*(1-\rho_q)\hat{q}_t - p_h^* y^*(1-\rho_y)\hat{y}_t \qquad \text{(A.3)}$$

It is straightforward to show that equation (A.3) has two roots, one of which is greater than one and the other of which is less than one.[23] Without losing generality, let η denote the root that is less than one. Solving equation (A.3) by forward iterations then shows that the equilibrium international bond holdings level is determined by the following cross-equation restriction (CER):

$$\begin{aligned}\tilde{b}_t &= \eta\tilde{b}_{t-1} + \beta\eta\lambda p_h^* y^* \sum_{j=0}^{\infty}(\beta\eta)^j E_t\hat{a}_{t+j} + \beta\eta p_h^* y^*(1-\rho_y)\sum_{j=0}^{\infty}(\beta\eta)^j E_t\hat{y}_{t+j} - \beta\eta p_h^* y^*(1-\rho_q)\sum_{j=0}^{\infty}(\beta\eta)^j E_t\hat{q}_{t+j},\\ &= \eta\tilde{b}_{t-1} + \frac{\beta\eta\lambda p_h^* y^*}{1-\beta\eta(1-\lambda)}\hat{a}_t + \frac{\beta\eta p_h^* y^*(1-\rho_y)}{1-\beta\eta\rho_y}\hat{y}_t - \frac{\beta\eta p_h^* y^*(1-\rho_q)}{1-\beta\eta\rho_q}\hat{q}_t. \qquad \text{(A.4)}\end{aligned}$$

[23] To characterize the roots of the second-order expectational difference equation, see, for example, Sargent (1987).

Substituting equation (A.4) back into equation (A.2) provides the CER for the exchange rate (36):

$$\hat{s}_t = \frac{\beta\eta - 1}{\beta p_h^* y^*}\tilde{b}_{t-1} + \frac{\beta\eta\lambda}{1-\beta\eta(1-\lambda)}\hat{a}_t + \frac{1-\kappa}{1-\kappa\rho_m}\hat{m}_t - \frac{1-\kappa}{1-\kappa\rho_\phi}\hat{\phi}_t - \frac{1-\beta\eta}{1-\beta\eta\rho_y}\hat{y}_t + \frac{1-\beta\eta}{1-\beta\eta\rho_q}\hat{q}_t + \frac{\kappa\rho_M}{1-\kappa\rho_M}\hat{\gamma}_{M,t}.$$

Therefore, in this symmetric case, the competitive equilibrium along the balanced growth path is characterized by a lower dimensional dynamic system of $(\hat{s}_t, \tilde{b}_t, \hat{a}_t, \hat{\gamma}_{M,t}, \hat{m}_t, \hat{\phi}_t, \hat{y}_t, \hat{q}_t)$.

Adding the log-linearized home and foreign budget constraints together implies the resource constraint $\hat{c}_{h,t} + \hat{c}_{f,t} = \hat{y}_{h,t} + \hat{y}_{f,t}$. Since the equilibrium dynamics of the consumption differential follow $\hat{c}_{h,t} - \hat{c}_{f,t} = -\hat{s}_t + \hat{q}_t + \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} + (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t - (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$, the home country's consumption obeys $2\hat{c}_{h,t} = (\hat{y}_{h,t}+\hat{y}_{f,t}) - \hat{s}_t + \hat{q}_t + \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} + (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t - (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$, while the foreign country's is $2\hat{c}_{f,t} = (\hat{y}_{h,t}+\hat{y}_{f,t}) + \hat{s}_t - \hat{q}_t - \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} - (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t + (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$. The home country's price $\hat{p}_{h,t}$ then is determined as follows. The Euler equation and the money demand function of the foreign country, (26) and (28), imply the expectational difference equation of $\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t}$

$$\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t = \kappa E_t(\hat{s}_{t+1} - \hat{p}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1} - \hat{\gamma}^f_{M,t+1}) - (1-\kappa)(\hat{m}_{f,t} - \hat{\phi}_{f,t}).$$

Solving the above equation by forward iterations and imposing a suitable transversality condition yields the CER $\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t = -\kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}^f_{M,t} - (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_{f,t} + (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_{f,t}$. This CER characterizes the equilibrium home price

$$2\hat{p}_{h,t} = \hat{s}_t - (\hat{y}_{h,t} + \hat{y}_{f,t}) - \hat{q}_t + \frac{\kappa\rho_M}{1-\kappa\rho_M}(\hat{\gamma}^h_{M,t} + \hat{\gamma}^f_{M,t}) + \frac{1-\kappa}{1-\kappa\rho_m}(\hat{m}_{h,t} + \hat{m}_{f,t}) - \frac{1-\kappa}{1-\kappa\rho_\phi}(\hat{\phi}_{h,t} + \hat{\phi}_{f,t}).$$

The money demand functions of both countries, (24) and (28), imply that the interest rates in the two countries are

$$(1+\hat{r}^h_{h,t}) = (1-\kappa)\left(\frac{\rho_M}{1-\kappa\rho_M}\hat{\gamma}^h_{M,t} - \frac{1-\rho_m}{1-\kappa\rho_m}\hat{m}_{h,t} + \frac{1-\rho_\phi}{1-\kappa\rho_\phi}\hat{\phi}_{h,t}\right)$$
$$(1+\hat{r}^f_{w,t}) = (1-\kappa)\left(\frac{\rho_M}{1-\kappa\rho_M}\hat{\gamma}^f_{M,t} - \frac{1-\rho_m}{1-\kappa\rho_m}\hat{m}_{f,t} + \frac{1-\rho_\phi}{1-\kappa\rho_\phi}\hat{\phi}_{f,t}\right).$$

Finally, as the last endogenous variable, the world interest rate of the home bonds then fluctuates in response to the risk premium, following $(1+\hat{r}^h_{w,t}) = (1+\hat{r}_{h,t}) + \psi(1-\kappa)\tilde{b}_t$.

Suppose that $\psi = 0$: There is no debt elastic risk premium in the home country's interest rate. It is easy to show that in this case, the second-order expectational difference equation (A.3) has a unit root, i.e., $\eta = 1$, and the resulting forward solution turns out to be

$$\tilde{b}_t = \tilde{b}_{t-1} + \frac{\beta\lambda p_h^* y^*}{1-\beta(1-\lambda)}\hat{a}_t + \frac{\beta p_h^* y^*(1-\rho_y)}{1-\beta\rho_y}\hat{y}_t - \frac{\beta p_h^* y^*(1-\rho_q)}{1-\beta\rho_q}\hat{q}_t.$$

Hence, the stochastic process of the de-trended international bond holding $\tilde{b}_t$ contains a permanent unit root component and never converges to the steady state. This lack of stationarity of the equilibrium balance growth path motivates this paper to allow for a positive elasticity of the risk premium with respect to the debt level.

Importantly, a permanent stochastic process of the de-trended international bond holding also emerges even when $\kappa = 1$. Because the log-linearized home country's interest rates (29) imply that under $\kappa = 1$, the debt elastic risk premia in play no role in determining the interest rates faced by the home country. As a result, the de-trended international bond holding $\tilde{b}_t$ contains a permanent unit root component, as in the case where $\psi = 0$. Hence, the closing of the two-country DSGE model in this paper requires the market discount factor to be strictly less than one.

Appendix B. Derivation of the error correction representation (33)

Let n_t denote the fundamental of the DSGE-PVM (31): $n_t \equiv \ln M_t - \ln C_t - \psi\kappa\tilde{b}_t - \ln\phi_t + \ln q_t$. Consider the currency return $\Delta \ln S_t$ adjusted by the fundamental $(1-\kappa)n_{t-1}$: $\Delta \ln S_t + (1-\kappa)n_{t-1}$. The DSGE-PVM (31) then implies:

$$
\begin{aligned}
\Delta \ln S_t + (1-\kappa)n_{t-1} &= (1-\kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i} + (1-\kappa)\sum_{j=0}^{\infty}\kappa^j E_{t-1}n_{t+i} \\
&\quad - (1-\kappa)\sum_{j=0}^{\infty}\kappa^j E_{t-1}n_{t+i-1} + (1-\kappa)n_{t-1}, \\
&= (1-\kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i} + \frac{(1-\kappa)^2}{\kappa}\sum_{i=0}^{\infty}\kappa^i E_{t-1}n_{t+i-1} - \frac{(1-\kappa)^2}{\kappa}n_{t-1}, \\
&= (1-\kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i} + \frac{1-\kappa}{\kappa}\ln S_{t-1} - \frac{(1-\kappa)^2}{\kappa}n_{t-1}.
\end{aligned}
$$

This result means that the currency return has the following error correction representation, given by equation (33):

$$
\begin{aligned}
\Delta \ln S_t = \frac{1-\kappa}{\kappa}&(\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} + \psi\kappa\tilde{b}_{t-1} + \ln\phi_{t-1} - \ln q_{t-1}) \\
&+ (1-\kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i}.
\end{aligned}
$$

Appendix C. Data description and construction

All data for the United States are distributed by Federal Reserve Economic Data (FRED), operated by the Federal Reserve Bank of St. Louis (http://http://research.stlouisfed.org/fred2/). The consumption data are constructed as the sum of the real personal consumption expenditures on non-durables and services. FRED, however, distributes only the nominal values of the two categories of personal consumption expenditures as *Personal Consumption Expenditure on Non-Durables (PCND)* and *Personal Consumption Expenditure on Services (PCESV)*. To construct the real total personal consumption expenditure $C_{us,t}$, I first calculate the share of the two nominal consumption categories in the nominal total personal consumption expenditure *Personal Consumption Expenditure* and then multiply the real total personal consumption expenditures, *Real Personal Consumption Expenditures at Chained 2005 Dollars (PCECC96)*, by the calculated share. Following NR, I adopt the M1 money stock, *M1SL*, as the aggregate money supply $M_{us,t}$. The nominal interest rate $r_{us,t}$ is provided by *three-month Treasury Bill (TB3MS)*. All the variables except the nominal interest rate are seasonally adjusted at annual rates and converted to the corresponding per capita terms by *Total Population (POP)*.

All Canadian data are distributed by Statistics Canada (CANSIM) (http://www5.statcan.gc.ca/cansim/). The real consumption data $C_{can,t}$ are constructed as the sum of *Personal Expenditure on Non-Durables at Chained 2002 Dollars*, *Personal Expenditure on Semi-Durables at Chained 2002 Dollars*, and *Personal Expenditure on Services at Chained 2002 Dollars*. I use the M1 money stock as the money supply $M_{can,t}$. The nominal interest rate $r_{can,t}$ is provided by *three-Month Treasury Bills*. All the variables except the nominal interest rate are seasonally adjusted at annual rates and converted to the corresponding per capita terms by *Estimate of Total Population*.

The output measures for Canada and the United States, $Y_{can,t}$ and $Y_{us,t}$, are constructed as in a model-consistent way. In this two-country endowment economy model, a country 's output is given by the sum of consumption and the trade balance. To measure the bilateral trade balance between Canada and the United States, TB_t, I use the Canadian goods trade balance for the United States included in CANSIM's balance of international payments data (CANSIM Table 376-0005). The Canadian output $Y_{can,t}$ is constructed by $C_{can,t} + TB_t$ and the United States output $Y_{us,t}$ is constructed by $C_{us,t} - TB_t/S_t$, where S_t is the bilateral exchange rate between Canada and the United States.

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Table 1: Prior Distributions of Structural Parameters

	Parameters	Distribution	Mean	S.D.	95 % Coverage
β	Household Subjective Discount Factor	Uniform(0,1)	—	—	[0.025 0.975]
γ_M	Deterministic (Gross) Money Growth	Gamma	1.015	0.005	[1.005 1.024]
ψ	Debt Elasticity of Risk Premium	Gamma	0.010	0.001	[0.008 0.012]
λ	Technology Diffusion Speed	Beta	0.010	0.001	[0.008 0.012]
ϕ	Mean Money Demand Shock	Gamma	1.000	0.010	[0.981 1.019]
ρ_M	Permanent Money Growth AR(1) Coef.	Beta	0.100	0.010	[0.081 0.120]
ρ_q	RER AR(1) Coef.	Beta	0.850	0.050	[0.739 0.933]
ρ_ϕ	Money Demand AR(1) Coef.	Beta	0.900	0.100	[0.633 0.998]

Note 1. The AR(1) coefficients of the transitory money and output shocks, ρ_m and ρ_y respectively, have the mass points of zero for identification.

Note 2. The standard deviations of all the structural shocks, σ_M, σ_A, σ_m, σ_y, σ_q, σ_ϕ, have the identical inverse Gamma prior distribution, with a mean of 0.01 and standard deviation of 0.01 for the benchmark information set.

Note 3.: The prior distribution of β is given by the Beta distribution, with a mean of 0.999 and standard deviation of 0.001 for the High Discount Factor model.

Table 2: Posterior Distributions of Structural Parameters

	Benchmark				HDF			
Parameters	Mean	S.D.	90 % Interval		Mean	S.D.	90 % Interval	
κ	0.612	0.002	[0.608	0.615]	0.926	0.003	[0.920	0.931]
β	0.622	0.002	[0.619	0.626]	0.998	0.001	[0.998	0.999]
γ_M	1.016	0.003	[1.009	1.021]	1.078	0.003	[1.072	1.084]
ψ	0.010	0.001	[0.008	0.012]	0.011	0.001	[0.009	0.013]
λ	0.009	0.001	[0.008	0.011]	0.006	0.000	[0.005	0.007]
ϕ	0.993	0.002	[0.990	0.997]	0.999	0.006	[0.989	1.008]
ρ_M	0.096	0.003	[0.091	0.099]	0.096	0.003	[0.091	0.100]
ρ_q	0.919	0.002	[0.916	0.923]	0.901	0.005	[0.894	0.909]
ρ_ϕ	0.996	0.001	[0.994	0.998]	0.987	0.001	[0.984	0.990]
σ_M	0.018	0.001	[0.015	0.019]	0.034	0.002	[0.031	0.038]
σ_A	0.006	0.000	[0.005	0.006]	0.006	0.000	[0.005	0.006]
σ_m	0.009	0.001	[0.008	0.010]	0.064	0.002	[0.059	0.067]
σ_y	0.003	0.000	[0.002	0.003]	0.003	0.003	[0.002	0.003]
σ_q	0.007	0.001	[0.006	0.009]	0.007	0.000	[0.006	0.007]
σ_ϕ	0.027	0.002	[0.024	0.030]	0.047	0.002	[0.042	0.050]
Marginal Likelihood	2145.267				1787.058			

Note 1: The "Benchmark" represents the Benchmark specification of the two-country model and the "HDF" represents the High Discount Factor specification.

Note 2: The marginal likelihoods are estimated based on Geweke's (1999) harmonic mean estimator.

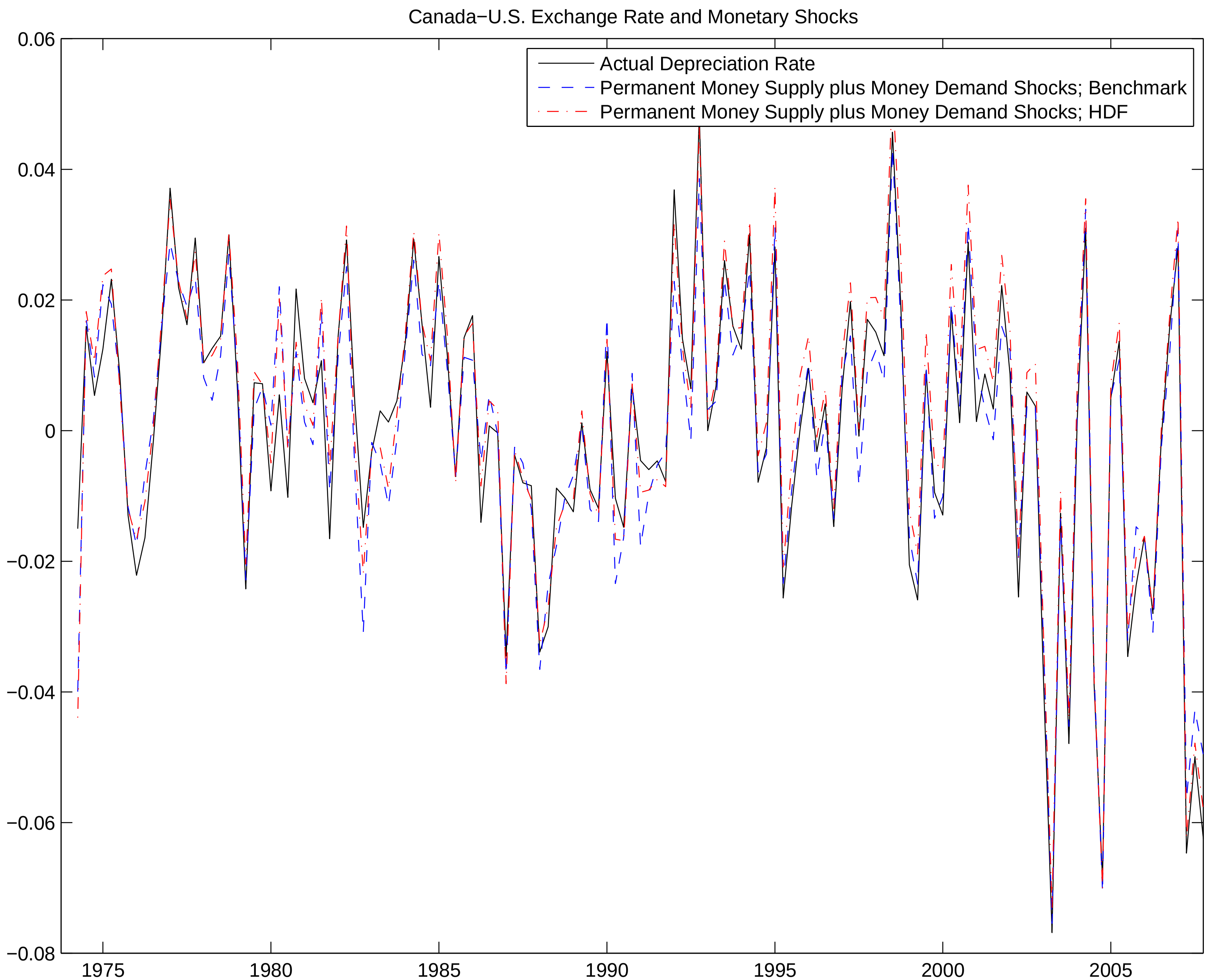

Figure 1: Depreciation rates and smoothed monetary shocks. Note: The solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. The dashed blue and dotted red lines respectively indicate the sum of the Kalman smoothers of the permanent money supply differential shock $\epsilon_{M,t}$ and the money demand differential shock $\epsilon_{\phi,t}$ for the Benchmark and HDF models. The Kalman smoothers are evaluated at the posterior means of the structural parameters.

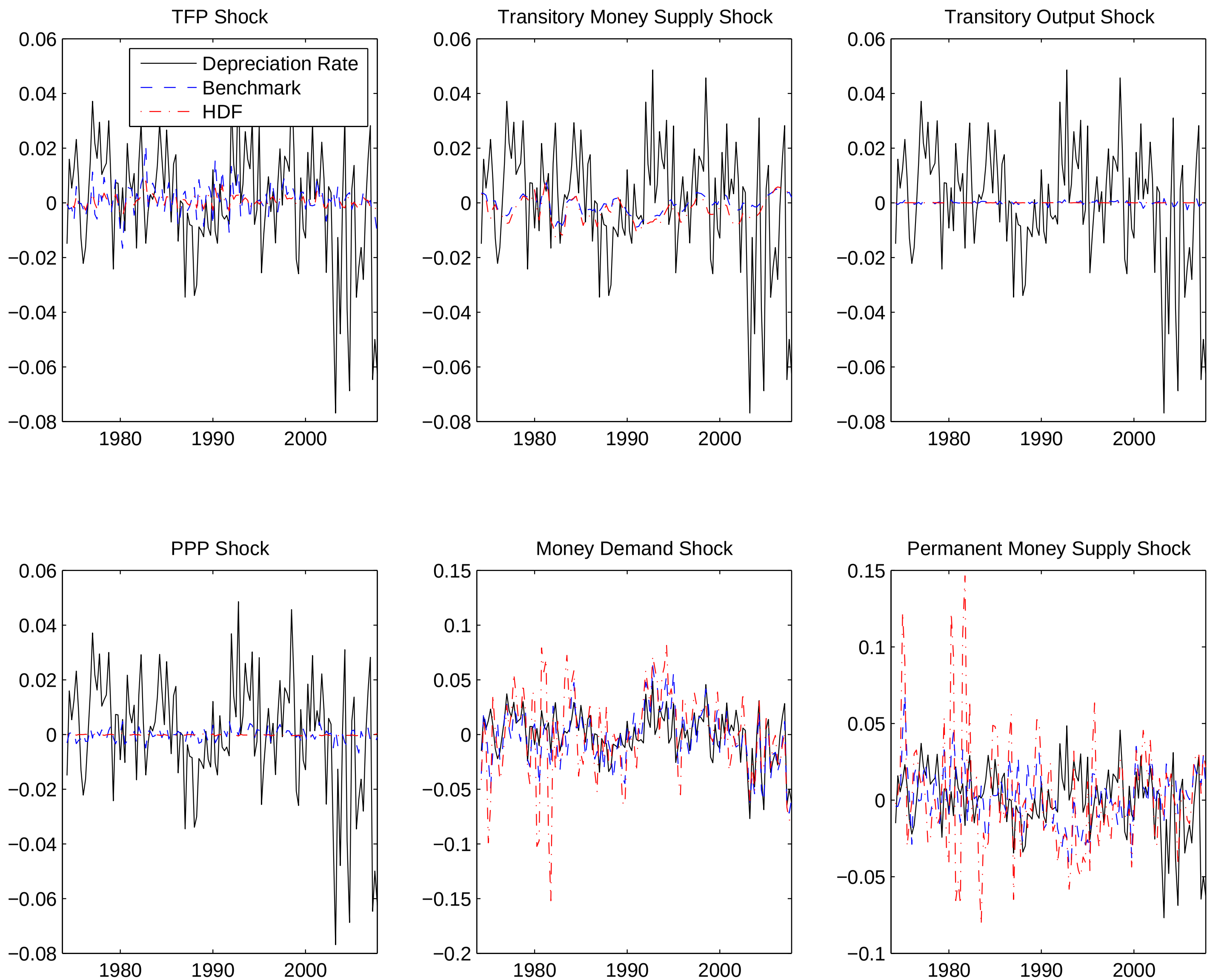

Figure 2: Depreciation rates and impact effects of structural shocks. Note: In each window, the solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. In each window with a particular structural shock, the dashed blue and dotted red lines, respectively, represent the Kalman smoother of the impact effect of the corresponding structural shock on the depreciation rate for the Benchmark and HDF models. The Kalman smoothers are evaluated at the posterior mean of the structural parameters. The calculation of the impact effects of the structural shocks is explained in detail in section 4.3.

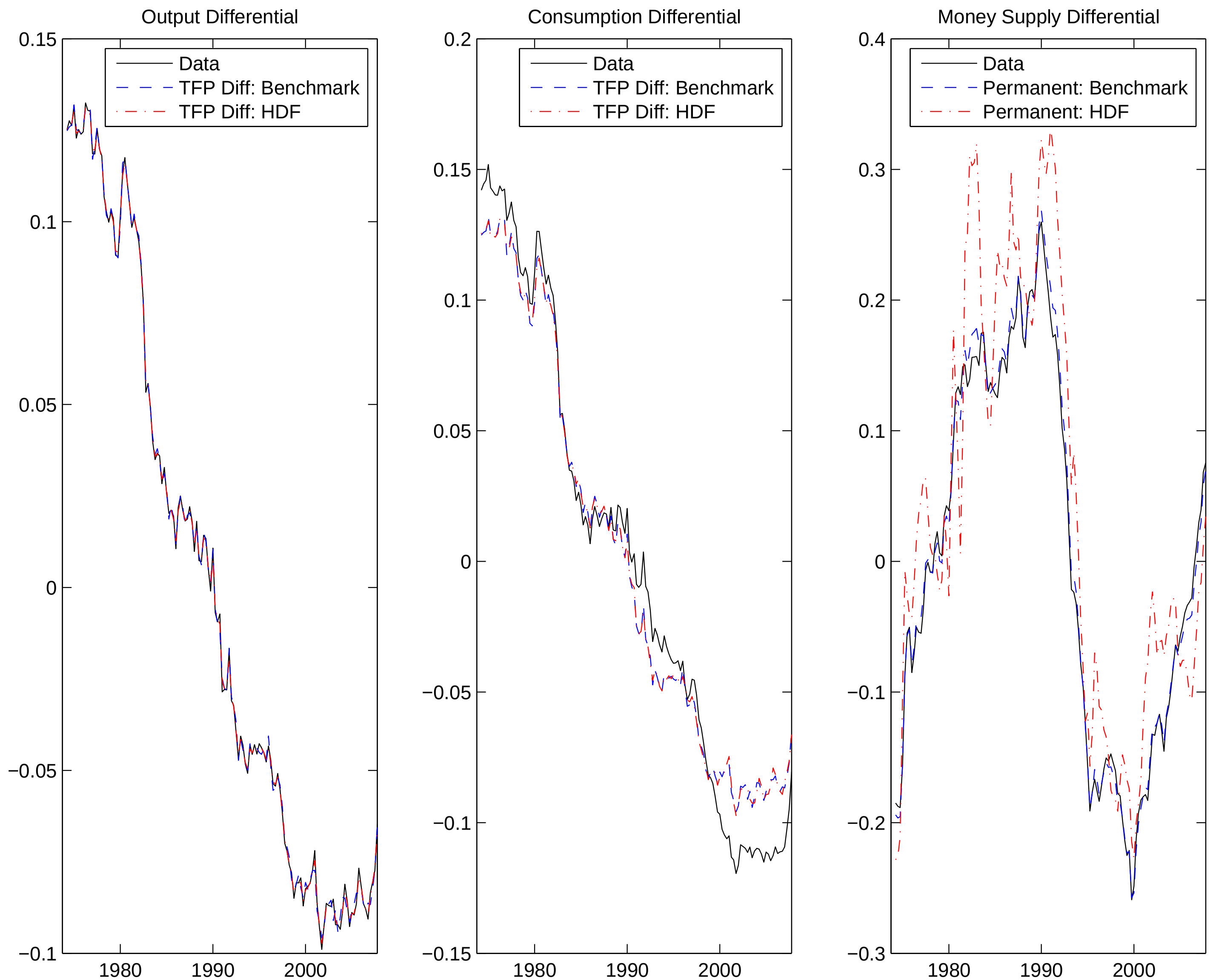

Figure 3: Identification of slow-moving trend components of cross-country differential data. Note. In the left window, the solid black line represents the actual log output differential and the dashed blue and dotted red lines indicate the Kalman smoothed TFP differential $\ln A_t$ of the Benchmark and HDF models, respectively. In the middle window, the black solid line represents the actual log consumption differential and the blue and red dashed lines represent the Kalman smoothed TFP differential $\ln A_t$ of the Benchmark and HDF models, respectively. In the right window, the black solid line represents the actual log money supply differential and the blue and red dashed lines indicate the Kalman smoothed permanent money supply differential $\ln M_t^\tau$ of the Benchmark and HDF models, respectively.

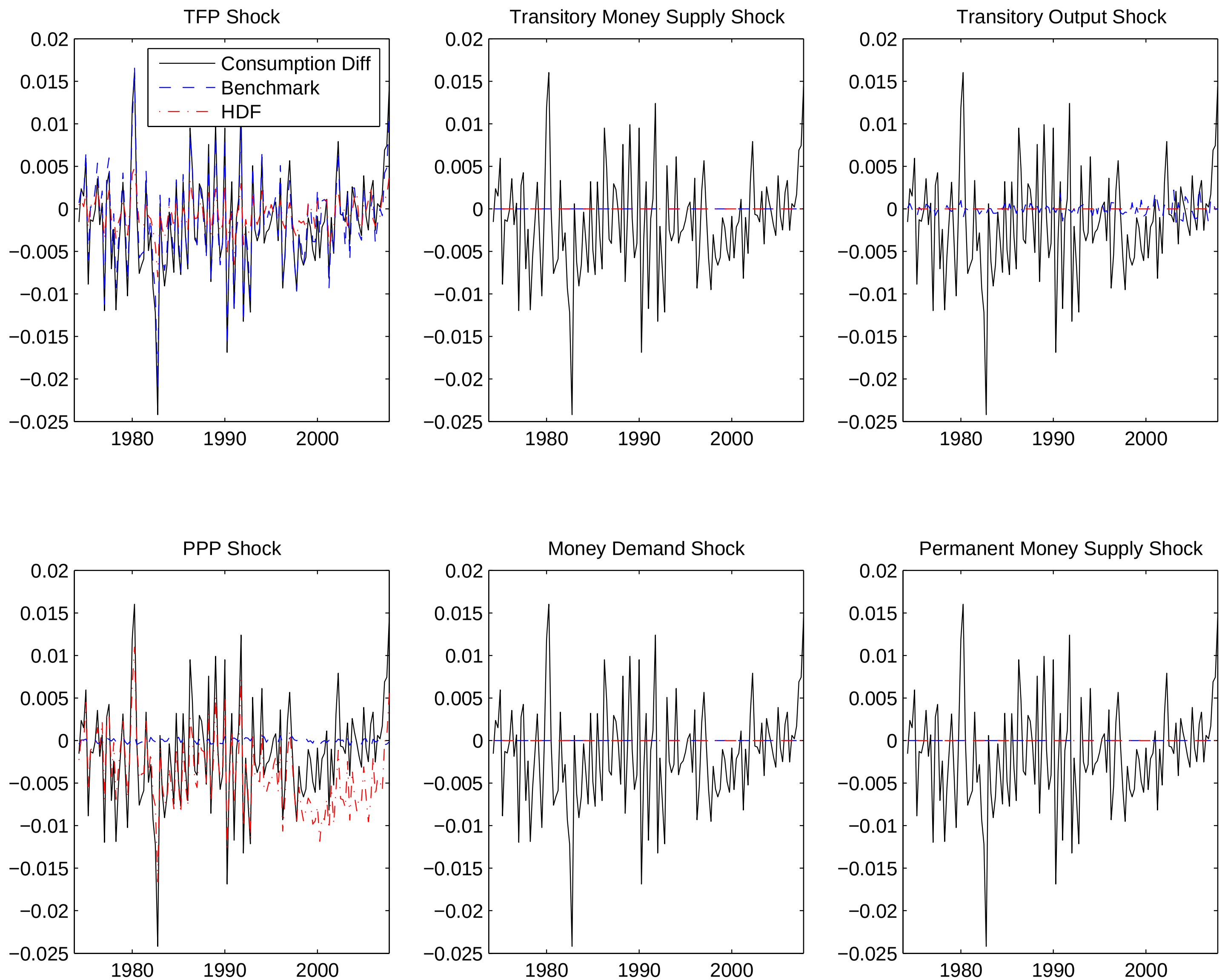

Figure 4: Consumption differential change and impact effects of structural shocks. Note: In each window, the solid black line represents the actual change in the log consumption differential. In each window with a particular structural shock, the dashed blue and dotted red lines, respectively, indicate the Kalman smoother of the impact effect of the corresponding structural shock on the depreciation rate for the Benchmark and HDF models. The Kalman smoothers are evaluated at the posterior mean of the structural parameters. The calculation of the impact effects of the structural shocks is explained in detail in section 4.3.

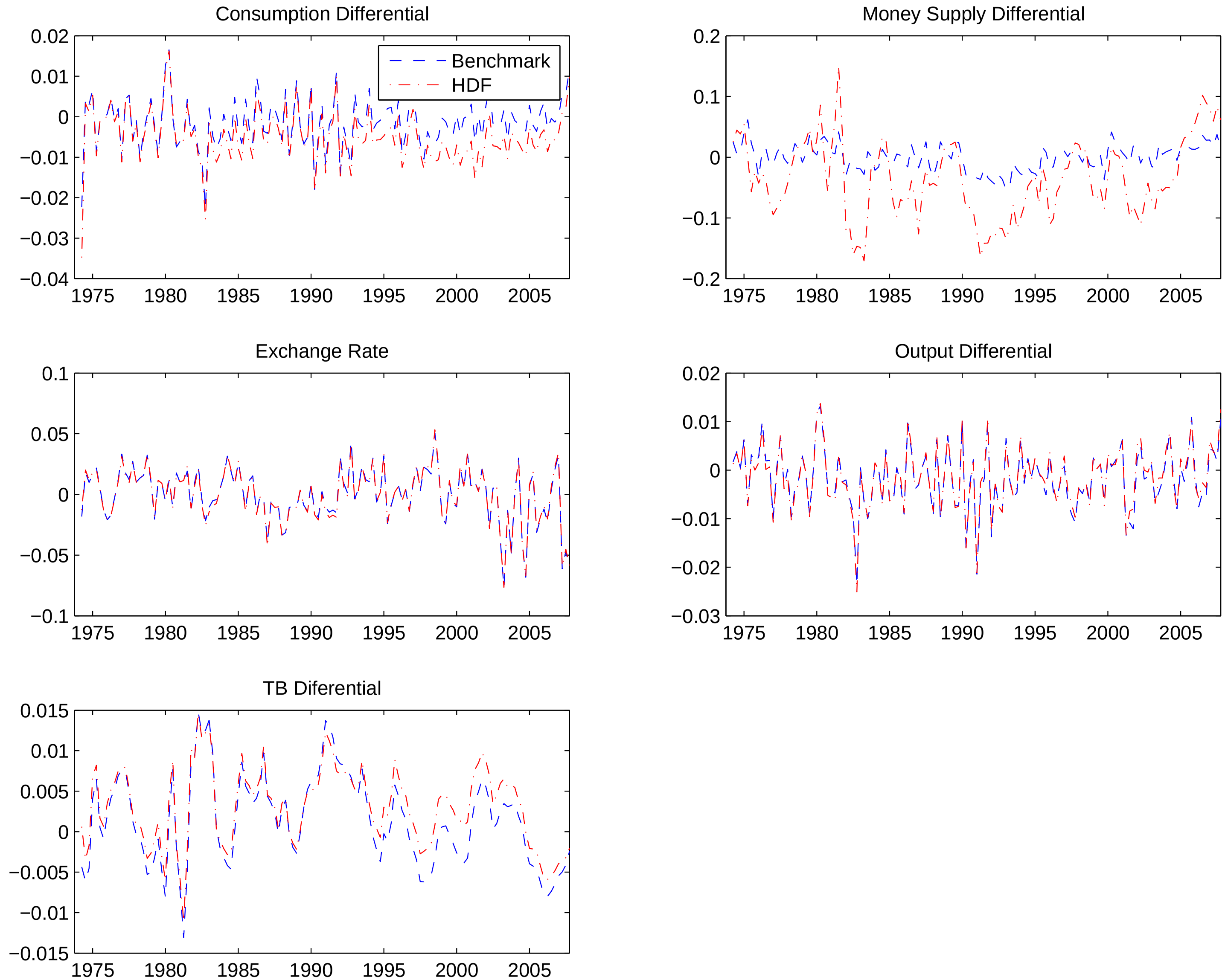

Figure 5: Forecast errors for observations. Note: In each window corresponding to a particular observation, the dashed blue and dotted red lines display the one-period-ahead forecast error calculated through the Kalman forward recursion based of the state space representation of the Benchmark and HDF models, respectively.